18 Exotic Options

Answers to Questions and Problems

1. Using the following parameter values, find the price of a forward-start put: $S = 100$; $X = 100$; $T - t = 1$ year; $\sigma = 0.2$; $r = 0.1$; $\delta = 0.05$; $tg = 0.5$.

   The first step is to value a plain vanilla put on the grant date, which is in one-half year. At that time, the put will have a half year remaining until expiration. Therefore,

$$d_1^M = \frac{\ln\left(\frac{100}{100}\right) + [0.1 - 0.05 + 0.5(0.2)(0.2)](0.5)}{0.2\sqrt{0.5}} = 0.247487$$

$$d_2^M = 0.247487 - 0.141421 = 0.106066$$

$$p_{tg}^M = Xe^{-r(T-t)}N(-d_2^M) - e^{-\delta(T-t)}S_tN(-d_1^M)$$

$$= 100e^{-.1(0.5)}N(-0.106066) - e^{-0.05(0.5)}100\,N(-0.247487)$$

$$= 100e^{-.1(0.5)}0.457765 - e^{-0.05(0.5)}100(0.402266)$$

$$= 4.3106$$

   Forward-Start Put $= e^{-\delta(tg-t)}p_{tg}^M = e^{-.05(.5)}4.3106 = 4.2042$

2. Price all four types of compound options assuming the following parameter values: $S = 100$; $\sigma = 0.4$; $r = 0.1$; $\delta = 0.05$; $X = 100$; $x = 8$; $T = 1$ year; $te = 0.25$ years.

   The first, and most difficult, step is to find the critical prices for the underlying calls and puts. For underlying calls, we have:

$$S^*e^{-\delta(T-te)}N(z) - Xe^{-r(T-te)}N\left(z - \sigma\sqrt{T-te}\right) - x = 0$$

   For underlying puts, the critical stock price satisfies the following relationship:

$$S^*e^{-\delta(T-te)}N(-z) + Xe^{-r(T-te)}N\left(-z + \sigma\sqrt{T-te}\right) - x = 0$$

   where:

$$z = \frac{\ln\left(\frac{S^*}{X}\right) + (r - \delta + 0.5\sigma^2)(T-te)}{\sigma\sqrt{T-te}}$$

These critical stock prices must be found by an iterative search. The critical stock price for an underlying call is 86.6162, and for an underlying put the critical stock price is 110.1995.These values can be verified as follows, first for underlying calls:

$$z = \frac{\ln\left(\frac{86.6152}{100}\right) + (0.1 - 0.05 + 0.5(0.4)(0.4))(0.75)}{0.4\sqrt{0.75}} = -0.133353$$

Therefore,

$$N(z) = -0.133353; \quad N\left(z - \sigma\sqrt{T - te}\right) = -0.479763$$

Thus, we verify that 86.6152 is the correct critical stock price for underlying calls:

$$86.6152\, e^{\,0.05\,(0.75)}\, 0.446957 - 100\, e^{\,0.1\,(0.75)}\, 0.315698 - 8 = 0$$

The same verification can be performed for underlying puts.

Before computing the actual option values, we must compute other intermediate results:

$$w_1 = \frac{\ln\left(\frac{S}{S^*}\right) + (r - \delta + 0.5\sigma^2)(te - t)}{\sigma\sqrt{te - t}}$$

$$w_2 = \frac{\ln\left(\frac{S}{X}\right) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

With our sample values, $w_1 = 0.880974$ for underlying calls, and $w_1 = -0.323111$ for underlying puts. The other intermediate variables are invariant across all option types: $w_2 = 0.325000$; $\rho = 0.5$. We also need the following bivariate cumulative normal probabilities for compound options on underlying calls:

$$N_2(w_1; w_2; \rho) = N_2(0.880974; 0.325; 0.5) = 0.564506$$

$$N_2\left(w_1 - \sigma\sqrt{te - t};\, w_2 - \sigma\sqrt{T - t};\, \rho\right) = N_2(0.680974; -0.075; 0.5) = 0.416949$$

$$N_2(-w_1; w_2; -\rho) = N_2(-0.880974; 0.325; -0.5) = 0.062904$$

$$N_2\left(-w_1 + \sigma\sqrt{te - t};\, w_2 - \sigma\sqrt{T - t};\, -\rho\right) = N_2(-0.680974; -0.075; -0.5) = 0.053158$$

$$N\left(w_1 - \sigma\sqrt{te - t}\right) = N(0.680974) = 0.752056$$

$$N\left(-w_1 + \sigma\sqrt{te - t}\right) = N(-0.680974) = 0.247944$$

For convenience, we also note:

$$Se^{-\delta(T - t)} = 100\, e^{-0.05\,(1)} = 95.122942$$

$$Xe^{-r(T - t)} = 100\, e^{-0.1\,(1)} = 90.483742$$

$$xe^{-r(te - t)} = 8\, e^{-0.1\,(0.25)} = 7.802479$$

With all of these preliminary calculations out of the way, we are ready to compute the price of compound options on underlying calls from our pricing formulas:

$$CC_t = Se^{-\delta(T-t)} N_2(w_1; w_2; \rho) - Xe^{-r(T-t)} N_2\left(w_1 - \sigma\sqrt{te-t}; w_2 - \sigma\sqrt{T-t}; \rho\right)$$
$$-xe^{-r(te-t)} N\left(w_1 - \sigma\sqrt{te-t}\right)$$

$$PC_t = -Se^{-\delta(T-t)} N_2(-w_1; w_2; -\rho) + Xe^{-r(T-t)} N_2\left(-w_1 + \sigma\sqrt{te-t}; w_2 - \sigma\sqrt{T-t}; -\rho\right)$$
$$+xe^{-r(te-t)} N\left(-w_1 + \sigma\sqrt{te-t}\right)$$

$$CC_t = 95.122942(0.564506) - 90.483742(0.416949) - 7.802479(0.752056)$$
$$= 10.1024$$

$$PC_t = -95.122942(0.062904) + 90.483742(0.053158) + 7.802479(0.247944)$$
$$= 0.7608$$

We now turn to the pricing of compound options on underlying puts. Earlier we noted that the critical price is 110.1995 and the appropriate value for w_1 is –0.323111. The probabilities that we need for the underlying puts are:

$$N_2(w_1; -w_2; \rho) = N_2(0.323111; -0.325; 0.5)$$
$$= 0.305486$$

$$N_2\left(-w_1 + \sigma\sqrt{te-t}; -w_2 + \sigma\sqrt{T-t}; \rho\right) = N_2(0.523111; 0.075; 0.5)$$
$$= 0.442964$$

$$N_2(w_1; -w_2; -\rho) = N_2(-0.323111; -0.325; -0.5)$$
$$= 0.067104$$

$$N_2\left(w_1 - \sigma\sqrt{te-t}; -w_2 + \sigma\sqrt{T-t}; -\rho\right) = N_2(-0.523111; 0.075; -0.5)$$
$$= 0.086930$$

$$N\left(-w_1 + \sigma\sqrt{te-t}\right) = N(0.523111)$$
$$= 0.699552$$

$$N\left(w_1 - \sigma\sqrt{te-t}\right) = N(-0.523111)$$
$$= 0.300448$$

We can now apply our formulas to compute the values of the compound options on underlying puts.

$$CP_t = -Se^{-\delta(T-t)} N_2(-w_1; -w_2; \rho) + e^{-r(T-t)} N_2\left(-w_1 + \sigma\sqrt{te-t}; -w_2 + \sigma\sqrt{T-t}; \rho\right)$$
$$-xe^{-r(te-t)} N\left(-w_1 + \sigma\sqrt{te-t}\right)$$

$$PP_t = Se^{-\delta(T-t)} N_2(+w_1; -w_2; \rho) - e^{-r(T-t)} N_2\left(w_1 - \sigma\sqrt{te-t}; -w_2 + \sigma\sqrt{T-t}; -\rho\right)$$
$$+xe^{-r(te-t)} N\left(w_1 - \sigma\sqrt{te-t}\right)$$

$$CP_t = -95.122942(0.305486) + 90.483742(0.442964) - 7.802479(0.699552) = 5.5641$$

$$PP_t = 95.122942(0.067104) - 90.483742(0.086930) + 7.802479(0.300448) = 0.8617$$

3. Price a simple chooser option based on the following parameter values: $S = 100$; $X = 100$; $T - t = 1$ year; $\sigma = 0.5$; $r = 0.1$; $\delta = 0.05$; $tc = 0.5$ years. By comparing this result with that of the example chooser in the sample text, what can you conclude about the influence of the stock's risk on the value of the chooser?

To price the chooser, we must first compute the following parameters:

$$w_1 = \frac{\ln\left(\frac{S}{X}\right) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + (0.1 - 0.05 + 0.5(0.5)(0.5))(1.0)}{0.5\sqrt{1.0}}$$

$$= 0.35$$

$$w_2 = \frac{\ln\left(\frac{S}{X}\right) + (r - \delta)(T - t) + 0.5\sigma^2(tc - t)}{\sigma\sqrt{tc - t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + (0.1 - 0.05)(1.0) + 0.5(0.5)(0.5)(0.5)}{0.5\sqrt{0.5}}$$

$$= 0.318198$$

$$N(w_1) = N(0.35) = 0.636831$$

$$N(-w_2) = N(-0.318198) = 0.375167$$

$$N\left(w_1 - \sigma\sqrt{T - t}\right) = N(0.35 - 0.5) = 0.440382$$

$$N\left(-w_2 + \sigma\sqrt{tc - t}\right) = N(-0.318198 + 0.353553) = 0.514102$$

Noting for convenience:

$$Se^{-\delta(T-t)} = 100\,e^{\;0.05(1.0)} = 95.122942$$

$$Xe^{-r(T-t)} = 100\,e^{-0.1(1.0)} = 90.483742$$

We now compute the value of the chooser according to our valuation formula:

$$Chooser_t = Se^{-\delta(T-t)}N(w_1) - Xe^{-r(T-t)}N(w_1 - \sigma\sqrt{T-t})$$

$$+Xe^{-r(T-t)}N\left(-w_2 + \sigma\sqrt{tc - t}\right) - Se^{-\delta(T-t)}N(-w_2)$$

$$Chooser_t = (95.122942)(0.636831) - 90.483742(0.440382)$$

$$+90.483742(0.514102) - 95.122942(0.375167)$$

$$= 31.5606$$

The higher the volatility, the greater the value of the chooser option.

4. Find the value of a down-and-in put with: $S = 100$; $X = 100$; $(T - t) = 1$ year; $\sigma = 0.3$; $r = 0.1$; $\delta = 0.05$; BARR = 97; and REBATE = 2.

We begin by computing a host of intermediate values:

$$\lambda = \frac{r - \delta + 0.5\sigma^2}{\sigma^2} = 1.055556$$

$$w_2 = \frac{\ln\left(\frac{S}{\text{BARR}}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t} = 0.418197$$

$$w_3 = \frac{\ln\left(\frac{\text{BARR}^2}{SX}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t} = 0.113605$$

$$w_4 = \frac{\ln\left(\frac{\text{BARR}}{S}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t} = 0.215136$$

$$N\left(-w_2 + \sigma\sqrt{T-t}\right) = N(-0.118197) = 0.452956$$

$$N(-w_2) = N(-0.418197) = 0.337901$$

$$N\left(w_4 - \sigma\sqrt{T-t}\right) = N(-0.084864) = 0.466185$$

$$N(w_4) = N(0.215136) = 0.585169$$

$$N\left(w_3 - \sigma\sqrt{T-t}\right) = N(-0.186395) = 0.426067$$

$$N(w_3) = N(0.113605) = 0.545225$$

$$N\left(w_2 - \sigma\sqrt{T-t}\right) = N(0.118197) = 0.547044$$

$$Xe^{-r(T-t)} = 90.483742$$

$$Se^{-\delta(T-t)} = 95.122942$$

$$\left(\frac{\text{BARR}}{S}\right)^{2\lambda} = 0.937721$$

$$\left(\frac{\text{BARR}}{S}\right)^{2\lambda-2} = 0.996621$$

$$\text{REBATE}\, e^{-r(T-t)} = 1.809675$$

Using the equations from Table 18.2:

DP2 = 90.483742 (0.452956) − 95.122942 (0.337901) = 8.843017
DP3 = 90.483742 (0.996621) (0.466185) − 95.122942 (0.937721) (0.585169)
= −10.156731
DP4 = 90.483742 (0.996621) (0.426067) − 95.122942 (0.937721) (0.545225)
= −10.211536
DP5 = 1.809675 [0.547044 − 0.996621 (0.466185)] = 0.149179

Finally, if $X >$ BARR, then from Table 18.5:

$$\begin{aligned}\text{DIP} &= \text{DP2} + \text{DP3} - \text{D4} + \text{DP5} \\ &= 8.843017 + 10.156731 + 10.211536 + 0.149179 \\ &= 9.0468\end{aligned}$$

5. Consider a cash-or-nothing call and put, with common parameter values: $S = 100$; $X = 110$; $T - t = 0.5$ years; $\sigma = 0.4$; $r = 0.1$; $\delta = 0.0$; and $Z = 200$. What is the value of each option? What is the value of a long position in both options? Which items of information given above are not needed to value the portfolio of the two options?

As the first step, we need to determine:

$$d_2^M = \frac{\ln\left(\frac{S_t}{X}\right) + (r - \delta + .5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} - \sigma\sqrt{T - t}$$

$$= \frac{\ln\left(\frac{100}{110}\right) + [0.1 - 0.0 + 0.5(0.4)(0.4)](0.5)}{0.4\sqrt{0.5}} - 0.4\sqrt{0.5}$$

$$= -0.301617$$

$N(d_2^M) = N(-0.301617) = 0.3481472$ and $N(-d_2^M) = N(0.301617) = 0.618528$. Therefore,

$$\text{CONC}_t = Ze^{-r(T-t)} N(d_2^M) = 200\, e^{-0.1(0.5)}\,(0.381472) = 72.5735$$

$$\text{CONP}_t = Ze^{-r(T-t)} N(-d_2^M) = 200\, e^{-0.1(0.5)}\,(0.618528) = 117.64724$$

The value of d_2^M and the associated probabilities are not necessary to value a portfolio of a cash-or-nothing call and put. The portfolio will pay the cash amount Z at expiration, so the portfolio must be worth the present value of Z at all times:

$$\text{CONC}_t + \text{CONP}_t = Ze^{-r(T-t)} = 72.5735 + 117.6724 = 190.2459$$

6. Consider an asset-or-nothing call and put, with common parameter values: $S = 100$; $X = 110$; $T - t = 0.5$ years; $\sigma = 0.4$; $r = 0.1$; and $\delta = 0.0$. What is the value of each option? What is the value of a long position in both options? Which items of information given above are not needed to value the portfolio of the two options?

First, we compute the value of d_1^M and its associated probabilities:

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + (r - \delta + .5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{110}\right) + (0.1 - 0.0 + .5(0.4)(0.4))(0.5)}{0.4\sqrt{0.5}}$$

$$= -0.018774$$

$N(d_1^M) = 0.492511$ and $N(-d_1^M) = 0.507490$. Thus,

$$\text{AONC}_t = e^{-\delta(T-t)} S_t N(d_1^M) = e^{-0.0(0.5)}\, 100\,(0.492511) = 49.2510$$

$$\text{AONP}_t = e^{-\delta(T-t)} S_t N(-d_1^M) = e^{-0.0(0.5)}\, 100\,(0.507484) = 50.7490$$

The value of a portfolio of a call and put will pay the asset at expiration, thus the value of the portfolio is:

$$\text{AONC}_t + \text{AONP}_t = e^{-\delta(T-t)} S_t = e^{-0.0(0.5)}\, 100 = 100$$

Therefore, valuing the portfolio does not require knowledge of d_1^M and its associated probabilities.

7. Value a gap call with: $S = 100$; $X = 100$; $T - t = 0.5$ years; $\sigma = 0.5$; $r = 0.1$; $\delta = 0.03$; and $g = 7$.

The value of a gap call is:

$$\text{GAPC}_t = e^{-\delta(T-t)} S_t N(d_1^M) - (X + g)\, e^{-r(T-t)} N(d_2^M)$$

Therefore, we first compute d_1^M, d_2^M, and their associated probabilities:

$$d_1^M = \frac{\ln\left(\frac{100}{100}\right) + [0.1 - 0.03 + 0.5(0.5)(0.5)]\,(0.5)}{0.5\sqrt{0.5}} = 0.275772$$

$$d_2^M = d_1^M - \sigma\sqrt{T-t} = 0.275772 - 0.5\sqrt{0.5} = -\,0.077782$$

$N(d_1^M) = N(0.275772) = 0.608638$, and $N(d_2^M) = N(-0.077782) = 0.469001$. Therefore,

$$\begin{aligned}\text{GAPC}_t &= e^{-0.03\,(0.5)}\,100(0.608638) - (100 + 7)\, e^{-0.1(0.5)}\,(0.469001)\\ &= 12.2221\end{aligned}$$

8. Value a supershare with: $S = 100$; $X_L = 95$; $X_H = 110$; $T - t = 0.5$ years; $\sigma = 0.2$; $r = 0.1$; $\delta = 0.05$. By comparing this calculation with the sample supershare of the text, what can you conclude about the value of supershares and the value $X_H - X_L$?

The valuation formula, shown below, requires computing w_L and w_H.

$$SS = \frac{Se^{-\delta(T-t)}}{X_L}[N(w_L) - N(w_H)]$$

$$L = \frac{\ln\left(\frac{S}{X_L}\right) + (r - \delta + 0.5\sigma^2)\,(T + t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{95}\right) + [0.1 - 0.05 + 0.5(0.2)(0.2)]\,0.5}{0.2\sqrt{0.5}}$$

$$= 0.610186$$

and:

$$w_H = \frac{\ln\left(\frac{S}{X_H}\right) + (r - \delta + 0.5\sigma^2)\,(T - t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{110}\right) + [0.1 - 0.05 + 0.5\,(0.2)(0.2)]\,(0.5)}{0.2\sqrt{0.5}}$$

$$= -0.426457$$

$N(0.610186) = 0.729131$, and $N(-0.426457) = 0.334887$.

$$SS = \frac{100e^{-0.05(0.5)}}{95}[0.729131 - 0.334887] = 0.4047$$

For the text example and this problem, all parameters are the same except for the payoff range. In the text example, the payoff range ran from 100 to 105 with a supershare value of 0.1332, while in this problem it runs from 95 to 105, with a supershare value of 0.4047. Thus, the larger the payoff range, the greater the value of the supershare. This only makes sense, because the broader the payoff range, the greater the chance that the supershare will finish in-the-money. Notice that it is not only the size of the range, but its location. For instance, if the range ran from 0 to 25, the bounds would be wider, but this supershare would be worth zero.

9. Find the value of a lookback call and put with the common parameters: $S = 110$; $T - t = 1$ year; $\sigma = 0.25$; $r = 0.08$; and $\delta = 0.0$. For the call, MINPRI = 80. For the put, MAXPRI = 130.

Computing the value of a lookback call requires the following intermediate values:

$$b = \ln\left(\frac{S}{\text{MINPRI}}\right) = \ln\left(\frac{110}{80}\right) = 0.318454$$

$$\mu = r - \delta - 0.5\sigma^2 = 0.08 - 0.0 - 0.5\,(0.25)\,(0.25) = 0.048750$$

$$\lambda = \frac{0.5\sigma^2}{r - \delta} = \frac{0.5\,(0.25)\,(0.25)}{0.08 - 0.0} = 0.390625$$

$$Se^{-\delta(T-t)} = 110e^{-0.0(1)} = 110$$

$$\text{MINPRI}\, e^{-r(T-t)} = 80\, e^{-0.008(1.0)} = 73.849308$$

$$N\left(\frac{b + \mu\,(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{0.318454 + 0.048750(1.0)}{0.25\sqrt{1.0}}\right) = 0.929058$$

$$N\left(\frac{-b + \mu\,(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{-0.318454 + 0.048750(1.0)}{0.25\sqrt{1.0}}\right) = 0.140335$$

$$N\left(\frac{-b - \mu\,(T-t) - \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{-0.318454 - 0.048750(1.0) - (0.25)\,(0.25)\,(1.0)}{0.25\sqrt{1.0}}\right) = 0.042824$$

The valuation formula for a lookback call is:

$$\begin{aligned}\text{LBC} = {} & Se^{-\delta(T-t)} - \text{MINPRI}\, e^{-r(T-t)} N\left(\frac{b + \mu\,(T-t)}{\sigma\sqrt{T-t}}\right) \\ & + \text{MINPRI}\, e^{-r(T-t)} \lambda\, e^{b(1-1/\lambda)} N\left(\frac{-b + \mu\,(T-t)}{\sigma\sqrt{T-t}}\right) \\ & - Se^{-\delta(T-t)}(1 + \lambda)\, N\left(\frac{-b - \mu\,(T-t) - \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\end{aligned}$$

Applying this to our values gives:

$$\begin{aligned}\text{LBC} = {} & 110 - 73.849308\,(.929058) \\ & + 73.849308(0.237688)(0.140335) - 110\,(1.390625)(0.042824) \\ = {} & 37.3023\end{aligned}$$

To value the lookback put, we use many of the same intermediate values. However, for the lookback put, the b term is different:

$$b = \ln\left(\frac{S}{\text{MAXPRI}}\right) = \ln\left(\frac{110}{130}\right) = -0.167054$$

The cumulative normal values are:

$$N\left(\frac{b-\mu(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{-0.167054 - 0.048750(1.0)}{0.25\sqrt{1.0}}\right)$$
$$= 0.681971$$

$$N\left(\frac{b-\mu(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{-0.167054 - 0.048750(1.0)}{0.25\sqrt{1.0}}\right)$$
$$= 0.194009$$

$$N\left(\frac{b+\mu(T-t)+\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) = N\left(\frac{-0.167054 + 0.048750(1.0) + (0.25)(0.25)(1.0)}{0.25\sqrt{1.0}}\right)$$
$$= -0.223216$$

We also note that:

$$\text{MAXPRI}e^{-r(T-t)} = 120.005125$$

The valuation formula for the lookback put is:

$$\text{LBP} = -Se^{-\delta(T-t)} + \text{MAXPRI}e^{-r(T-t)}N\left(\frac{-b-\mu(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$-\text{MAXPRI}e^{-r(T-t)}\lambda\, e^{b(1-1/\lambda)}N\left(\frac{b-\mu(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$+Se^{-\delta(t-t)}(1+\lambda)N\left(\frac{b+\mu(T+t)+\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

With our values, we have:

$$\text{LBP} = -110 + 120.005125\,(0.681971)$$
$$-120.005125\,(0.506920)\,(0.194009) + 110\,(1.390625)\,(0.411683)$$
$$= -110 + 81.840015 - 11.802140 + 62.974634$$
$$= 23.0125$$

10. Find the value of an average price option with these common parameters: $S = 100$; $X = 90$; $\sigma = 0.2$; $r = 0.1$; $\delta = 0.05$; $t_0 = 0.0$; $t_1 = 0.5$; $t_2 = 0.5$; and $A = 95$. Compute the value of the option with observations every two days, $h = 2/365$. Now compute the value of the option assuming continuous observation, that is, $h = 0$. Compare these results with the sample option of the chapter. What does this suggest about the value of the option and the frequency of observation?

$$W = A^{[t_1/(t_1+t_2+h)]}S^{[(t_2+h)/(t_1+t_2+h)]}$$
$$= 95^{(0.5/0.5+0.5+0.005479)}100^{(0.5+0.005479/0.5+0.5+0.005479)}$$
$$= 97.481565$$

$$M = \left(t_0 + t_2 \frac{t_2 + h}{2(t_1 + t_2 + h)} \right) [r - \delta - 0.5\sigma^2]$$

$$= \left(0.0 + 0.5 \frac{0.5 + 0.005479}{2(0.5 + 0.5 + 0.005479)} \right) [0.1 - 0.05 - 0.5\,(0.2)\,(0.2)]$$

$$= 0.003770$$

$$\Sigma^2 = t_0 + \left(\frac{t_2\,(t_2 + h)\,(2t_2 + h)}{6(t_1 + t_2 + h)^2} \right) \sigma^2$$

$$= 0.0 + \left(\frac{0.5(0.5 + 0.005497)\,[2(0.5) + 0.005497]}{6(0.5 + 0.5 + 0.005497)^2} \right) (0.2)\,(0.2)$$

$$= 0.001676$$

$$w_1 = \frac{\ln\left(\frac{W}{X}\right) + M}{\Sigma} + \Sigma$$

$$= \frac{\ln\left(\frac{97.481565}{90}\right) + 0.003770}{0.040936} + 0.040936$$

$$= 2.083730$$

We note that $N(w_1) = 0.981408$, and $N(w_1 - \Sigma) = 0.979464$. The valuation formula for an average price option is:

$$\text{AVGPRI} = We^{-r(T-t)}\, e^{(M + 0.5\Sigma 2)}\, N(w_1) - Xe^{-r(T-t)}\, N(w_1 - \Sigma)$$

The price of our example option is:

$$\text{AVGPRI} = (97.481565)\, e^{-0.1(0.5)}\, e^{[0.003770 + 0.5\,(0.001676)]}\, (0.981408)$$

$$-90\, e^{-0.1(0.5)}\, (0.979464)$$

$$= 91.423703 - 83.852548$$

$$= 7.5711$$

We now compute the price of the same option with $h = 0$.

$$W = A^{[t_1/(t_1 + t_2 + h)]} S^{[(t_2 + h)/(t_1 + t_2 + h)]}$$

$$= 95^{(0.5/0.5 - 0.5)} 100^{(0.5/0.5 + 0.5)}$$

$$= 97.46794$$

$$M = \left(t_0 + t_2 \frac{t_2 + h}{2(t_1 + t_2 + h)} \right) [r - \delta - 0.5\sigma^2]$$

$$= \left(0.0 + 0.5 \frac{0.5}{2(0.5 + 0.5)} \right) [0.1 - 0.05 - 0.5(0.2)(0.2)]$$

$$= 0.003750$$

$$\Sigma^2 = t_0 + \left(\frac{t_2(t_2 + h)(2t_2 + h)}{6(t_1 + t_2 + h)^2}\right)\sigma^2$$

$$= 0.0 + \left(\frac{0.5(0.5)[2(0.5)]}{6(0.5 + 0.5)^2}\right)(0.2)(0.2)$$

$$= 0.001667$$

$$w_1 = \frac{\ln\left(\frac{W}{X}\right) + M}{\Sigma} + \Sigma$$

$$= \frac{\ln\left(\frac{97.46794}{90}\right) + 0.003750}{0.040829} + 0.040829$$

$$= 2.085264$$

$N(w_1) = 0.981477$, and $N(w_1 - \Sigma) = 0.979545$. The price of our example option with continuous observation is:

$$\text{AVGPRI} = (97.46794)\, e^{-0.1(0.5)}\, e^{[0.003750 + 0.5(0.001667)]}\,(0.981477)$$

$$- 90\, e^{-0.1(0.5)}\,(0.979545)$$

$$= 91.415066 - 83.859482$$

$$= 7.5556$$

For the example average price option of the text and these problems, we find that the value of the option is: 7.5556 with continuous observation; 7.5634 with observations every day; and 7.5711 with observations every second day. This suggests that the price of the average option varies inversely with the observation frequency, but that the effect is quite small.

11. Consider an exchange option with the following common parameter values: $S_1 = 100$; $S_2 = 100$; $\sigma_1 = 0.3$; $\sigma_2 = 0.2$; $\delta_1 = 0.05$; $\delta_2 = 0.05$; $T - t = 0.5$ years. Compute the value of this exchange option with $\rho = 0.0$ and $\rho = 0.7$. Compare your results with those for the sample exchange option in the chapter. What do these results suggest about the value of exchange options as a function of the correlation between the two assets?

We begin by computing intermediate values as follows:

$$\Sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\,\sigma_2$$

$$= 0.3(0.3) + 0.2(0.2) - 2(0.0)(0.3)(0.2)$$

$$= 0.13$$

$$w_1 = \frac{\ln\left(\frac{S_2}{S_1}\right) + (\delta_1 - \delta_2 + 0.5\,\Sigma^2)(T - t)}{\Sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.05 - 0.05 + 0.5(0.13)](0.5)}{\sqrt{0.13}\,\sqrt{0.5}}$$

$$= 0.127475$$

$$w_2 = \frac{\ln\left(\frac{S_2}{S_1}\right) + (\delta_1 - \delta_2 + 0.5\,\Sigma^2)(T-t)}{\Sigma\sqrt{T-t}} - \Sigma\sqrt{T-t}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.05 - 0.05 + 0.5(0.13)](0.5)}{\sqrt{0.13}\sqrt{0.5}} - \sqrt{0.13}\sqrt{0.5}$$

$$= -0.127475$$

With these values, $N(w_1) = 0.550718$, and $N(w_2) = 0.449282$. The valuation equation is:

$$\text{EXOPT} = S_2\,e^{-\delta_2(T-t)}\,N(w_1) - S_1\,e^{-\delta_1(T-t)}\,N(w_2)$$

$$\text{EXOPT} = 100\,e^{-0.05(0.5)}\,0.550718 - 100\,e^{-0.05(0.5)}\,0.449282$$

$$= 53.712072 - 43.818919$$

$$= 9.8932$$

We now compute the value of the same option with $\rho = 0.7$.

$$\Sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\,\sigma_2$$

$$= 0.3(0.3) + 0.2(0.2) - 2(0.7)(0.3)(0.2)$$

$$= 0.046$$

$$w_1 = \frac{\ln\left(\frac{S_2}{S_1}\right) + (\delta_1 - \delta_2 + 0.5\,\Sigma^2)(T-t)}{\Sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.05 - 0.05 + 0.5(0.046)](0.5)}{\sqrt{0.046}\sqrt{0.5}}$$

$$= 0.075829$$

$$w_2 = \frac{\ln\left(\frac{S_2}{S_1}\right) + (\delta_1 - \delta_2 + 0.5\,\Sigma^2)(T-t)}{\Sigma\sqrt{T-t}} - \Sigma\sqrt{T-t}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.05 - 0.05 + 0.5(0.046)](0.5)}{\sqrt{0.046}\sqrt{0.5}} - \sqrt{0.046}\sqrt{0.5}$$

$$= -0.075829$$

With these values, $N(w_1) = 0.530223$, and $N(w_2) = 0.469777$. Therefore,

$$\text{EXOPT} = 100\,e^{-0.05(0.5)}\,0.530223 - 100\,e^{-0.05(0.5)}\,0.469777$$

$$= 51.713175 - 45.817134$$

$$= 5.8960$$

For these parameter values, but differing correlations, we have values as follows: for $\rho = 0.0$, the price is 9.8932; for $\rho = 0.5$, the price is 7.2687 as shown in the text; and for $\rho = 0.7$, the price is 5.8960. Therefore, the price of the option varies inversely with the correlation.

For rainbow options, consider these parameter values: $S_1 = 100$; $S_2 = 100$; $X = 95$; $T - t = 0.5$ years; $\sigma_1 = 0.4$; $\sigma_2 = 0.5$; $r = 0.06$; $\delta_1 = 0.02$; $\delta_2 = 0.03$; and $\rho = 0.2$. (Interpret $X = 95$ as the exercise price or as the cash payment depending on the type of option.) Use this information for problems 12–18.

All of the following problems rely on some common values that we compute here and use in the specific solutions below.

$$\Sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\,\rho\,\sigma_1\,\sigma_2 = 0.4(0.4) + 0.5(0.5) - 2(0.2)(0.4)(0.5) = 0.33$$

$$\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\Sigma} = \frac{0.2(0.5) - 0.4}{\sqrt{0.33}} = -0.522233$$

$$\rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\Sigma} = \frac{0.2(0.4) - 0.5}{\sqrt{0.33}} = -0.731126$$

$$w_1 = \frac{\ln\left(\frac{S_1}{X}\right) + (r - \delta_1 + 0.5\,\sigma_1^2)(T - t)}{\sigma_1\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{95}\right) + [0.06 - 0.02 + 0.5(0.4)(0.4)](0.5)}{0.4\sqrt{0.5}}$$

$$= 0.393481$$

$$w_2 = \frac{\ln\left(\frac{S_2}{X}\right) + (r - \delta_2 + 0.5\,\sigma_2^2)(T - t)}{\sigma_2\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{95}\right) + [0.06 - 0.03 + 0.5(0.5)(0.5)](0.5)}{0.5\sqrt{0.5}}$$

$$= 0.364282$$

$$w_3 = \frac{\ln\left(\frac{S_1}{S_2}\right) + (\delta_2 - \delta_1 + 0.5\,\Sigma^2)(T - t)}{\Sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.03 - 0.02 + 0.5(0.33)](0.5)}{\sqrt{0.33}\sqrt{0.5}}$$

$$= 0.215410$$

$$w_4 = \frac{\ln\left(\frac{S_2}{S_1}\right) + (\delta_1 - \delta_2 + 0.5\,\Sigma^2)(T - t)}{\Sigma\sqrt{T - t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [0.02 - 0.03 + 0.5(0.33)](0.5)}{\sqrt{0.33}\sqrt{0.5}}$$

$$= 0.190792$$

The basic evaluation units that we require are:

Q1. $S_1 e^{-\delta_1(T-t)} \{ N(w_3) - N_2(-w_1; w_3; \rho_1) \}$

Q2. $S_2 e^{-\delta_2(T-t)} \{ N(w_4) - N_2(-w_2; w_4; \rho_2) \}$

Q3. $Xe^{-r(T-t)} N_2\left(-w_1 + \sigma_1\sqrt{T-t}; -w_2 + \sigma_2\sqrt{T-t}; \rho\right)$

To evaluate Q1–Q3, we need the following intermediate values:

$$S_1 e^{-\delta_1(T-t)} = 100\, e^{-0.02(0.5)} = 99.004983$$

$$S_2 e^{-\delta_2(T-t)} = 100\, e^{-0.03(0.5)} = 98.511194$$

$$Xe^{-r(T-t)} = 95\, e^{-0.06(0.5)} = 92.192326$$

$$N(w_3) = N(0.215410) = 0.585276$$

$$N(w_4) = N(0.190792) = 0.575656$$

$$N_2(-w_1; w_3; \rho_1) = N_2(-0.393481; 0.215410; -0.522233) = 0.122795$$

$$N_2(-w_2; w_4; \rho_2) = N_2(-0.364282; 0.190792; -0.731126) = 0.084427$$

$$N_2\left(-w_1 + \sigma_1\sqrt{T-t}; -w_2 + \sigma_2\sqrt{T-t}; \rho\right) = N_2\left(-0.393481 + 0.4\sqrt{0.5}; -0.364282 + 0.5\sqrt{0.5}; 0.2\right) = 0.257875$$

Using the previous intermediate results, we now compute Q1–Q3:

$$Q1 = S_1 e^{-\delta_1(T-t)} \{N(w_3) - N_2(-w_1; w_3; \rho_1)\}$$
$$= 99.004983(0.585276 - 0.122795)$$
$$= 45.787924$$

$$Q2 = S_2 e^{-\delta_2(T-t)} \{N(w_4) - N_2(-w_2; w_4; \rho_2)\}$$
$$= 98.511194(0.575656 - 0.084427)$$
$$= 48.391555$$

$$Q3 = Xe^{-r(T-t)} N_2\left(-w_1 + \sigma_1\sqrt{T-t}; -w_2 + \sigma_2\sqrt{T-t}; \rho\right)$$
$$= 92.192326(0.257875)$$
$$= 23.774096$$

12. Find the value of an option on the best of two assets and cash.

$$\text{BEST3} = Q1 + Q2 + Q3 = 45.787924 + 48.391555 + 23.774096 = 117.9536$$

13. Find the value of an option on the better of two assets.

The valuation equation is CBETTER = BEST3, given that $X = 0$. If $X = 0$, several values computed above must be adjusted. First, Q3 = 0. However, w_1 and w_2 both change as well. Each has X alone in its denominator, thus w_1 and w_2 are both infinite. These are important in the computation of the bivariate cumulative normal values. For present purposes, within the context of the unit normal distribution, let these variables be set equal to 10.0. As shown below, the large values of w_1 and w_2 force the bivariate probabilities to zero.

$$Xe_{-r(T-t)} = 0\, e_{0.06(0.5)} = 0.0$$
$$N_2(-w_1; w_3; \rho_1) = N_2(-10.0; 0.215410; -0.522233) = 0.0$$
$$N_2(-w_2; w_4; \rho_2) = N_2(-10.0; 0.190792; -0.731126) = 0.0$$

These affect the computation of Q1 and Q2 as follows:

$$\begin{aligned} Q1 &= S_1 e^{-\delta_1(T-t)}\{N(w_3) - N_2(-w_1; w_3; \rho_1)\} \\ &= 99.004983\,(0.585276 - 0.0) \\ &= 57.945254 \\ Q2 &= S_2 e^{-\delta_2(T-t)}\{N(w_4) - N_2(-w_2; w_4; \rho_2)\} \\ &= 98.511194\,(0.575656 - 0.0) \\ &= 56.708521 \end{aligned}$$

Therefore, CBETTER = Q1 + Q2 + Q3 = 57.945254 + 56.708521 + 0 = 114.6538.

14. Find the value of a call on the maximum of two assets.

$$\text{CMAX} = \text{BEST3} - Xe^{-r(T-t)} = Q1 + Q2 + Q3 - Xe^{-r(T-t)} = 117.9536 - 92.192326 = 25.761249$$

15. Find the value of a put on the maximum of two assets.

$$\text{PMAX} = \text{CMAX} - \text{CBETTER} + Xe^{-r(T-t)} = 25.761249 - 114.6538 + 92.192326 = 3.2998$$

16. Find the value of a call on the minimum of two assets.

$$\text{CMIN} = C_t^M(S_1) + C_t^M(S_2) - \text{CMAX} = 14.4949 + 16.7708 - 25.761249 = 5.504451$$

17. Find the value of a call on the worse of two assets.

$$\begin{aligned} \text{CWORSE} &= C_t^M(S_1) + C_t^M(S_2) - \text{CBETTER, given that } X = 0 \\ &= 99.004983 + 98.511194 - 114.6538 = 82.8624 \end{aligned}$$

18. Find the value of a put on the minimum of two assets.

$$\text{PMIN} = \text{CMIN} - \text{CWORSE} + Xe^{-r(T-t)} = 5.504451 - 82.8624 + 92.192326 = 14.8344$$