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Alethic Modal Logics and Semantics

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1 Introduction

The first axiomatic development of modal logic was untertaken by C. I. Lewis in 1912. Being anticipated by H. McCall in 1880, Lewis tried to cure logic from the 'paradoxes' of *extensional* (i.e. *truthfunctional*) implication ⊃ (cf. Hughes and Cresswell 1968: 215). He introduced the stronger notion of *strict* implication <, which can be defined with help of a necessity operator □ (for 'it is neessary that:') as follows: A < B *iff* □(A ⊃ B); in words, A strictly implies B iff A necessarily implies B (A, B, . . . for arbitrary sentences). The new primitive sentential operator □ is *intensional* (*non-truthfunctional*): the truth value of A does *not* determine the truth-value of □A. To demonstrate this it suffices to find two particular sentences p, q which agree in their truth value without that □p and □q agree in their truth-value. For example, it p = 'the sun is identical with itself,' and q = 'the sun has nine planets,' then p and q are both true, □p is true, but □q is false. The dual of the necessity-operator is the possibility operator ◊ (for 'it is possible that:') defined as follows: ◊A *iff* ¬□¬A; in words, A is possible iff A's negation is not necessary. Alternatively, one might introduce ◊ as new primitive operator (this was Lewis' choice in 1918) and define □A as ¬◊¬A and A < B as ¬◊(A ∧ ¬B).

Lewis' work cumulated in Lewis and Langford (1932), where the five axiomatic systems S1–S5 were introduced. S1–S3 are weaker that the standard systems of § 2.2, but S4 and S5 coincide with standard S4 and S5 (for details on Lewis' systems cf. Hughes and Cresswell 1968: ch. 12; Chellas and Segerberg 1996). Lewis' pioneer work was mainly syntactic-axiomatic, except for the modal matrix-semantics (for details in the 'algebraic' tradition, started by Lukasiewicz, cf. Bull and Segerberg 1984: 8ff). The philosophically central semantics for modal logic is *possible world semantics.* It goes back to ideas of Leibniz, was first developed by Carnap and received broadest acceptance through the later work of Kripke. The *actual* world, in which we happen to live, is merely one among a multitude of other possible worlds, each realizing a different but logically complete collection of facts. The basic idea of possible world semantics as expressed by Carnap (1947: 9f, 174f) is:

(1) □A is true in the actual world iff A is true in all possible worlds.
◊A is true in the actual world iff A is true in some possible world.

Thus, the truth valuation of sentences is *relativized* to possible worlds (or just: *worlds*). In order to obtain a recursive definition, the truth of modalized sentences (e.g. □A, ◊A) must also be determined relative to possible worlds. Let W be a given set of worlds w, $w_1 \ldots \in$ W; then (1) is rephrased as follows:

(1*) □A [or: ◊A] is true in a given w ∈ W iff A is true in all [or some, resp.] w ∈ W

What is the *ontological status* of possible worlds? Forbes (1985: 74) distinguishes between three philosophical positions: (1) According to *absolute* realism, possible worlds exist and are entities sui generies. Lewis (1973: 84ff) has defended this position. (2) For *reductive* realism, possible worlds exist but can be reduced to more harmless (e.g. conceptual or linguistic) entities. (3) For *anti-realism*, possible worlds don't exist; so possible-world-sentences are either false or meaningless. While (1) and (3) are extreme positions, some variant of position (2) is the most common view. Kripke (1972: 15), for example, denies the 'telescope view' of worlds and conceives possible worlds as possible (counterfactual) *states* or *histories* of the actual world. (The 'possible state' – versus 'possible history' – interpretation is a further important choice; cf. Schurz 1997: 40f.) Those who still regard Kripke's counterfactual position as too problematic may alternatively conceive worlds as metalinguistic entities, namely as *interpretation functions* of the object language (this was Kripke's early view in 1959, whereas in 1963a he introduced W as a set of primitive objects). Even more scrupulous, Carnap (1947: 9) had identified worlds with object-language entities – his so-called *state descriptions*. Carnap's concept was generalized by Hintikka's (1961: 57–9) to so-called 'extended state descriptions' which in the terminology of the section below entitled "Axiomatic Systems: Correctness, Completeness, and Correspondence" are nothing but worlds of canonical models. *In the upshot:* possible world semantics does not force one into a particular metaphysical position.

A logically decisive but rarely discussed question is the determination of the set W of possible worlds. There are two options:

C-SEMANTICS We identify W with the *fixed* set $W^{\mathcal{L}}$ containing *all* worlds, or interpretations, which are logically possible in the given language $\mathcal{L}$. Then □ is a logically *constant* symbol with a *logically fixed* interpetation – that of *logical* necessity. A (modal or non-modal) sentence is then defined as *logically* true iff it is true in all worlds of $W^{\mathcal{L}}$.

K-SEMANTICS Alternatively, we consider W as a *varying* set of possible worlds, or interpretations, which need not comprise *all* logically possible worlds. Then □, though formally a *logical* symbol, has an implicitly varying interpretation (similar to ∀ in first-order logic because of the varying domain; cf. Schurz 1999). For example, if W contains all logically possible worlds, then □ means 'logically possible,' while if W contains only all physically possible worlds, then □ means 'physically possible.' In this setting, we count a sentence as logically true only if its truth does not depend on such special choices of W; thus we consider a (non-modal or modal) sentence as *logically* true iff it is true in all worlds w ∈ W *for all sets of possible worlds* W.

C-semantics is the semantics of Carnap (1946: 34; 1947: 9f, 174f). It leads to a modal logic which is called **C** in Schurz (2000). **C** is stronger than **S5** and exhibits non-classical features such as *failure of closure under substitution* (in **C**, $\Diamond p$ is a logically true for every propositional variable p, but $\Diamond(p \wedge \neg p)$ is logically false), or axiomatization by *non-monotonic* rules (if A is *not* a **C**-theorem, then $\Diamond\neg A$ is a **C**-theorem; cf. Schurz 2000 and Gottlob 1999). These are largely ignored facts, due to certain confusing historical peculiarities, for example that Carnap (1946) himself had announced to have obtained Lewis' system **S5**. But this result was based an *ad hoc* deviation: in his modal propositional logic, Carnap restricts the logical truths of **C** to the subclass of those formulas which are closed under substitution (1946: 40, D4-1) and shows that the so restricted class of theorems is equivalent to Lewis' **S5**. For further details see Hendry and Pokriefka (1985) and Schurz (2000), who defends **C** in spite of its non-classical features.

K-semantics has been introduced by Kripke (1959, 1963a, 1963b), whose papers have opened the highway to the modern modal logicians' industry. As proved by Kripke (1959), K-semantics leads exactly to the system **S5** (similar results were obtained by Hintikka (1961) and Kanger (1957a), not to forget Prior (1957), the founder of tense logic). K-semantics leads to modal logics which enjoy all classical properties of logics; on the cost that standard modal logics do not contain non-trivial possibility theorems (cf. Schurz 2000, theorems 3 + 4). Apart from this insufficiency, the theorems of **S5** are rather strong: for example for every *purely modal* sentence A (each atomic subformula of A occurs in the scope of a modal operator), $\Box A \vee \Box\neg A$ is **S5**-valid. The crucial step which utilized K-semantics for weaker systems and added an almost unlimited semantical flexibility to K-semantics was the introduction of the so-called *relation R of accessibility*, or 'relative possibility,' between possible worlds, independently by Kanger (1957a), Hintikka (1961) and Kripke (1963a). Thus, $w_1 R w_2$ means that world w_2 is accessible from (or possible with respect to) w_1, and the refined modal truth clause goes as follows:

(2) $\Box A$ is true in a given $w \in W$ iff A is true in *all* w^* such that wRw^*.
$\Diamond A$ is true in a given $w \in W$ iff A is true in *some* w^* such that wRw^*.

By varying structural conditions on the relation R (e.g. reflexive, transitive, symmetric) one gets different modal logics, among them the standard systems **T**, **S4**, and **S5**. A multitude of similar results were produced in the following decades, with outstanding modal logicians of the '2nd generation' such as Lemmon and Scott, Segerberg or Fine, to name just a few. While C-semantics was almost completeley neglected, K-semantics dominated the development of modal logic, whence the remaining sections focus on K-semantics. Due to K-semantical flexibility, various new philosophical interpretations of the modal operator have been discovered. For example, in systems weaker than **T**, the modal operator may be interpreted as 'it is obligatory that . . . ,' which leads to Kripkean semantics for so-called *deontic* logics, or as 'it is believed that. . . . ,' which brings us into *epistemic* logic, etc. (see Gabbay and Guenthner 1984, and ch. XIII of this volume). This development led to a broader understanding of 'modal logic' as the *logic of intensional propositional operators*, while the narrow meaning of modal logic as the logic of necessity and possibility is expressed in the specification '*alethic*' modal logic.

So far we have discussed only modal propositional logic – from now on: *MPL*. Many more difficulties are involved in *modal quantificational* (or predicate) *logic* – from now on: *MQL*. Here we have, besides W and R, a domain D of *individuals* (i.e. objects). Here we have two major choices.

CHOICE 1 Should we assume that singular terms denote the same object in all possible worlds (*rigid designators*), or that their reference object varies from world to world (*non-rigid designators*)?

CHOICE 2 Should we suppose that every object in D exists necessarily, that is exists in all possible worlds (*constant domain*), or should we better admit that some individuals may exist in one world without existing in another world (*varying domains*)?

Until today the difficulties connected with these choices have not been completely solved.

Quine's famous attack on the reasonableness of 'de re' modalities in 1943 started the well-documented debate on these choices (see Linsky 1971). A formula is called modally *de re* (in the 'strong' sense) iff an individual constant or variable in A occurs free in the scope of '□'; otherwise it is called *de dicto* (Fine 1978: 78, 135, 143; Forbes 1985: 48f). For example, □Fa and ∃x□Fx are *de re*, while □∃xFx is *de dicto*. The crucial semantical property of *de re* formulas is that their semantic evaluation requires an identification or correlation of objects across different worlds. For example, ∃x□Fx says that in our world there exists an individual which in all possible worlds has property F – in other words, F is an *essential* (i.e. necessary) properties of this individuals. Hence we assume that an object of our world – or at least some identifiable correlate of it – exists in all other possible worlds. In contrast, □∃xFx merely asserts that in all possible worlds *some* individual exists which has property F; this does not presuppose any correlation between individuals in different worlds. Thus, the semantical question of fixed versus varying domains and rigid versus nonrigid designators does not concern *de dicto* but only *de re* sentences.

Quine (1943) has argued that the reference of singular terms depends on *contingent* facts, whence modal contexts are opaque: substitution of identicals fails in them. In his famous example, both '□(9 > 7)' and '9 = the number of planets' are true, but '□(the number of planets > 7)' is obviously false. Quine concludes that modal *de re* statements lack clear meaning. Ruth Barcan-Marcus, who developed Lewis-style MQLs in 1946, gave a profound defence of MQL against Quine's attack. In 1960 Barcan-Marcus emphasized that the failure of substitution of identicals (a = b) ⊃ (A[a] ≡ A[b])) in MQLs does not deprive *de re* sentences from clear meaning. She also shows that substitution of necessary identicals (□(a = b) ⊃ (A[a] ≡ A[b])) still holds. In (1963), she argued that the reference of proper names – in contrast to definite descriptions – should indeed be regarded as the same across all possible worlds. In a modified form, this thesis was defended by Kripke (1972); he suggested the name 'rigid designator' and made it prominent, especially the connected thesis of *necessities aposteriori* (a = b ⊃ □(a = b); cf. 1972: 35–8).

Apart from Carnap's early work, the first semantically interpreted MQL-**S5** system was developed by Kripke (1959), who assumes rigid designators and constant domain.

Rigid designators are axiomatically reflected in the two theorems $(\neg)x = y \supset \Box(\neg)(x = y)$. Constant domains gets axiomatically cashed out in the Barcan formula BF: $\forall x\Box A \supset \Box\forall xA$ (introduced by Ruth Barcan 1946). Neither BF nor its converse cBF: $\Box\forall xA \supset \forall x\Box A$ are especially plausible; but varying domains cause drastic difficulties (see below, "Nonrigid designators, counterpart theory, and worldline semantics"). A comparatively simple system based on varying domains and rigid designators was developed by Kripke (1963b), on the cost of restricting necessitation rule. Hintikka (1961: 63f), argues in favor of varying domains and nonrigid designators (see also Hughes and Cresswell 1968: 190). Later, Lewis (1968) argued that individuals at different worlds can never be identical, but can merely be so-called *counterparts* of each other. His important philosophical point is that in order to avoid Quine's de re skepticism, it is not necessary to assume rigid designators; it suffices to assume the existence of a counterpart relation. To says that F is an *essential property* of a ($\Box Fa$) means in Lewis' theory that all counterparts of a in all possible worlds have property F (Lewis 1968: 118). Thus, although Kripke (1972) critizices Lewis, both agree in their *essentialism*, that is in their optimistic view about *de re* modalities.

The metaphysically significant alternative to both Kripke and Lewis is *de re skepticism*. The *de re* skeptics doubt that identifications or correlations of objects across possible world are an intelligible concept. Von Wright (1951: 26–8) suggested that in a satisfying modal logic all *de re* modalities should be eliminable in favor of *de dicto* modalities (see Hughes and Cresswell 1968: 184ff). This position was reconstructed as the position of 'anti-Haecceitism' by Fine (1978). According to its basic idea, the naming of individuals in possible worlds rests on purely conventional grounds. Thus, in an 'anti-Haecceitist' possible world model the accessible worlds should be closed under *local isomorphisms* w.r.t. their domains of individuals; Fine calls such possible world models *homogeneous* (1978: 283). A singular necessity statement $\Box Fa$ is true in a world w of a homogeneous model only if its universal closure $\forall x\Box Fx$ is true, too. Fine (1978: 281) proves that the quantificational system **S5 + H** is complete for the class of homogeneous possible world models. This system is obtained from **S5** by adding all $\forall$-$\Box$-closures of axiom H: $(\mathrm{Dif}(x_1, \ldots, x_n) \wedge \Box A) \supset \Box\forall x_1 \ldots \forall x_n(\mathrm{Dif}(x_1, \ldots, x_n) \supset A)$, where $\mathrm{Dif}(x_1, \ldots, x_n) =_{df} \wedge \{x_i = x_j: 1 \leq i < j \leq n\}$ and A's free variables are among $x_1, \ldots, x_n$.

2 Modal Propositional Logics (MPLs)

Language

In what follows, capital Latin A, B . . . will vary over formulas of the object language and capital Greek Γ, Δ, . . . over sets of them. F will always denote a frame and M a model, **F** a set of frames and **M** a set of models, W a possible world set, R the accessibility relation, and V a valuation function. The letters w, u, v will range over possible worlds (all symbols may also be used in an indexed way). We use all standard symbols of informal first-order logic and informal set theory, which forms our *metalanguage* (see van Dalen et al. 1978); in particular $\Rightarrow$ is the implication sign of the metalanguage. Our object language is $\mathcal{L}$, the language of MPL. It contains as nonlogical symbols a denumerably infinite set of propositional variables $\mathcal{P}$, and as primitive logical symbols the truth-functional connectives $\neg$ (negation), $\vee$ (disjunction) and the necessity operator $\Box$. The other truth-functional connectives $\wedge$ (conjunction), $\supset$ (material implication), $\equiv$ (material equivalence), $\top$ (Verum), $\bot$ (Falsum) and the possibility operator $\Diamond$ are defined as usual ($A \wedge B =_{df} \neg(\neg A \vee \neg B)$, $A \supset B =_{df} \neg A \vee B$, $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$, $\top =_{df} p \vee \neg p$, $\bot =_{df} p \wedge \neg p$, $\Diamond A =_{df} \neg \Box \neg A$). $\mathcal{L}$ is identified the set of its (well-formed) formulas, that is *sentences*, which are recursively defined as follows: (1) $p \in \mathcal{P} \Rightarrow p \in \mathcal{L}$, (2) $A \in \mathcal{L} \Rightarrow \neg A \in \mathcal{L}$, (3) $A, B \in \mathcal{L} \Rightarrow (A \vee B) \in \mathcal{L}$, (4) $A \in \mathcal{L} \Rightarrow \Box A \in \mathcal{L}$ (nothing else). $\mathcal{P}(A)$ = the set of propositional variables in A.

Possible Worlds Semantics

A *frame* is a pair $F = \langle W,R \rangle$ where $W \neq \emptyset$ (a nonempty set of 'possible worlds') and $R \subseteq W \times W$ (the accessibility relation; uRv abbreviates $\langle u,v \rangle \in R$). A *model* for $\mathcal{L}$ is triple $M = \langle W,R,V \rangle$ where $\langle W,R \rangle$ is a frame (we say that M is based on this frame) and $V: \mathcal{P} \to Pow(W)$ is a valuation function which assigns to each propositional variable $p \in \mathcal{P}$ the set of worlds $V(p) \subseteq W$ *at* which p is true ('Pow' for 'power set'). We also write W^F, R^F to indicate that W and R belong to F; and likewise for W^M, R^M and V^M. The assertion 'formula A is *true at* world w *in* model M' (where $w \in W^M$) is abbreviated as $(M,w) \models A$ and recursively defined as follows: (1) $(M,w) \models p$ iff $w \in V(p)$; (2) $(M,w) \models \neg A$ iff not $M \models A$, and $M \models A \vee B$ iff $M \models A$ or $M \models B$; finally (3) $(M,w) \models \Box A$ iff for all $u \in W$ such that wRu, $(M,u) \models A$. Sentence $A \in \mathcal{L}$ is defined as *valid in* model M, in short: $M \models A$, iff A is true at all worlds of M. The set of worlds verifying A in model M is also written as $\|A\|^M$ and considered as the *proposition* expressed by the sentence A in model M. Formula A is *valid on* a frame F, in short $F \models A$, iff A is valid in all models based on F. Formula A is valid w.r.t. (with respect to) a class of models $\mathbf{M}$, in short $\mathbf{M} \models A$, or w.r.t. a class of frames $\mathbf{F}$, in short $\mathbf{F} \models A$, iff A is valid in all $M \in \mathbf{M}$, or on all $F \in \mathbf{F}$, respectively. Analogously, a formula set Γ is valid in a model M, $M \models \Gamma$, iff all formulas in Γ are valid in M; analogously for validity of Γ on F, w.r.t. $\mathbf{M}$, and w.r.t. $\mathbf{F}$. A formula set $\Gamma \subseteq \mathcal{L}$ is said to be (simultaneously) *satisfiable* in a model M (or: w.r.t. a model-class $\mathbf{M}$) iff all formulas in Γ are true at some world in M (or: at some world in some $M \in \mathbf{M}$, respectively), and $\Gamma \subseteq \mathcal{L}$ is (simultaneously) *satisfiable* on a frame F (or: w.r.t. a frame-class $\mathbf{F}$) iff Γ is satisfiable on some model based on F (or: in some model based on some $F \in \mathbf{F}$).

Logics can be defined in a semantical way (this section) and in an axiomatic-syntactical way (next section). Let $\mathbf{M}(\mathbf{F})$ denote the class of all models based on some frame in frame-class $\mathbf{F}$, and call a model class $\mathbf{M}$ *frame-based* iff $\mathbf{M} = \mathbf{M}(\mathbf{F})$ for some $\mathbf{F}$. Frame classes are defined by purely structural conditions on R and allow all possible valuation functions. In contrast, not-frame-based model classes are defined by restrictions on the valuation function. A *logic*, however, should admit all possible valuations of its nonlogical symbols (see Schurz 1999). Therefore, frame-classes and frame-based model classes are the philosophically more important means to characterize modal logics, as compared to not-frame-based model-classes (such as the 'general frames' of cf. 'More metalogical results on PMLs' below). Semantically, a MPL $\mathbf{L}$ can be defined as the set of formulas which are valid w.r.t. a given class $\mathbf{F}$ of frames: $\mathbf{L} = \mathbf{L}(\mathbf{F}) = \{A: \mathbf{F} \models A\}$; the so-defined $\mathbf{L}$ is a 'normal' MPL. Formula A is said to be a valid consequence of

Γ w.r.t. frame class **F**, in short $\Gamma \models_{\mathbf{F}} A$, iff for all worlds w in all models M based on some frame in **F**, (M,w) $\models \Gamma$ implies (M,w) $\models$ A. If $\Gamma \models_{\mathbf{F}} A$, we also say that the *rule* Γ/A (read: 'Γ, therefore: A') is *valid* w.r.t. frame class **F**. Validity of a rule means *truth-preservation*. It is important to distinguish this from the *admissibility* of a rule, which means *validity-*preservation (Schurz 1994). In first-order logic, for example, Modus Ponens MP: A, A $\supset$ B/B is valid (truth-preserving) while Universal Generalization UG: A/$\forall$xA is merely admissible (validity preserving). We call a *rule* Γ/A (semantically) *admissible* w.r.t. frame class **F**, in short **F**-admissible, iff **F** $\models \Gamma$ implies **F** $\models$. A rule is called *frame-admissible* iff it preserves validity in every frame, and it is called *model-admissible* iff it preserves validity in every model. The reason for our definition of valid consequence (also called *local* consequence by van Benthem 1983: 37f) is that it implies the *Deduction Theorem:* Γ, A $\Vdash_{\mathbf{F}}$ B $\Rightarrow \Gamma \Vdash_{\mathbf{F}}$ A $\supset$ B. This theorem does not hold for merely frame-admissible consequences (which correspond to van Benthem's 'global' consequence).

With **FK** for the class of all (Kripke) frames, the following implication relation holds: Γ/A is **FK**-*valid* $\Rightarrow \Gamma/A$ is model-admissible $\Rightarrow \Gamma/A$ is frame-admissible $\Rightarrow \Gamma/A$ is **FK**-admissible. We first consider the logic **K** (for Kripke) which is semantically defined as the set of modal formulas which **FK**-valid, **K** = **L**(**FK**). Some terminology: A[B/C] denotes the result of replacing *some* occurrences of subformula B in A by C (so, strictly speaking, 'A[B/C]' varies over several formulas). A substitution function s:$\mathcal{P} \to \mathcal{L}$ substitutes arbitrary formulas s(p) for propositional letters p. The substitution instance s(A) results from A by replacing every p $\in \mathcal{P}$(A) in A by s(p).

***FK**-valid theorems*

Taut: Every tautology

K:	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	K◊:	$(\neg\Diamond A \wedge \Diamond B) \supset \Diamond(\neg A \wedge B)$
T:	$\Box \top$	T◊□:	$\neg\Diamond\bot$
C:	$(\Box A \wedge \Box B) \supset \Box(A \wedge B)$	C◊:	$\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
M:	$\Box(A \wedge B) \supset \Box A \wedge \Box B$	M◊:	$\Diamond A \vee \Diamond B \supset (\Diamond A \vee \Diamond B)$

Further theorems

1. $\Box A \vee \Box B \supset \Box(A \vee B)$,
2. $\Diamond(A \wedge B) \supset \Diamond A \wedge \Diamond B$,
3. $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$,
4. $\Box A \wedge \Diamond B \supset \Diamond(A \wedge B)$,
5. $\Box(A \vee B) \supset \Diamond A \vee \Box B$,
6. $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$,
7. $\Diamond(A \supset B) \equiv (\Box A \supset \Diamond B)$.

***FK**-valid rules*

TautR – all tautological rules in particular MP: A, A $\supset$ B/B.

Model-admissible rules

N: A/$\Box$B

E: A $\equiv$ B / $\Box$A $\equiv$ $\Box$B

RE: B $\equiv$ C / A $\equiv$ A[B/C]

Further

All rules resulting from valid theorems by applying deduction theorem

A frame-admissible rule

Subst: A/s(A) for every $s:\mathcal{P} \to \mathcal{L}$

PROOFS *Exercise* (see proof examples below). *Hints:* The tautological theorems and rules hold because the clauses for truth-functional connectives are the same as in non-modal logic. In other words, (classical) modal logics *contain* truth-functional logic. Rule RE ('replacing equivalents') is a consequence of E ('equivalence') and proved by induction on complexity of formulas. Rule N ('necessitation') and the principle K (Kripke) are characteristic for *normal* logics validated by Kripke frames, while rule E and principles M, C, and T are used to axiomatize the 'weaker' classical logics. Every □-theorem has a ◊-*dual* which obtained by replacing '□' by '¬◊¬' and applying tautological transformations. Note that □ distributes over ∧ in both directions (M, C), but □ distributes over ∨ only in one direction (i); thus □ behaves like an implicit universal quantifier. The same relations hold, dually, between ◊ and ∨ , ∧; so ◊ behaves like an implicit existential quantifier.

PROOF OF VALIDITY OF (K) We prove $\Vdash_{\mathbf{FK}} \Box(A \supset B) \supset (\Box A \supset \Box B)$ by assuming, for an arbitrary model M and world w in W^M, that (a): $(M,w) \models \Box(A \supset B)$, and (b): $(M,w) \models \Box A$, and by proving that (a) and (b) implies (c): $(M,w) \models \Box B$. By (a) and truth clauses, $(M,u) \models A$ implies $(M,u) \Vdash B$, for all u with wRu. By (b), $(M,u) \Vdash A$ holds for all u with wRu. Therefore, $(M,u) \models B$ holds for all u with wRu, which gives us (c). Q.E.D.

PROOF THAT N IS MODEL- (AND HENCE FRAME-) ADMISSIBLE By contraposition. Assume (for arbitrary M) that $M \not\models \Box A$. Then there exists $w \in W^M$ such that $(M,w) \not\models \Box A$ and hence $u \in W^M$ with wRu such that $(M,u) \not\models A$. So $M \not\models A$. Q.E.D.

Syntactical substitutions are semantically mirrored by corresponding variations of the valuation function. This is the content of the following *substitution lemma*: Define, for arbitrary substitution function s and valuation function V, $V_s(p) = V(s(p))$, for all $p \in \mathcal{P}$; and for given $M = \langle W,R,V\rangle$, let $M_s = \langle W,R,V_s\rangle$; thus M and M_s are based on the same frame. Then: For every $A \in \mathcal{L}$, M and $w \in W^M$: $(M,w) \models s(A)$ iff $(M_s,w) \models A$

PROOF *Exercise*: By induction on formula complexity; (see, for example, van Benthem 1983: 27, Lemma 2.5).

PROOF THAT (SUBST) IS FRAME-ADMISSIBLE By contraposition. Assume, for arbitrary $F = \langle W,R\rangle$ and s, that $F \not\models s(A)$. Thus there exists $M = \langle W,R,V\rangle$ based on F and $w \in W^M$ such that $(M,w) \not\models s(A)$. By *substitution lemma*, $(M_s,w) \not\models A$, where M_s is based on F. Thus $F \not\models A$. Q.E.D.

Closure under substitution is an important condition on logics. Expressing theorems as *schemata* (with schematic letters A, B, . . . ranging over arbitrary formulas) is a simple means of asserting that the theorems of a logic are closed under substitution.

For example, the set of all schematic instances of the formula schema $\Box A \supset A$ equals the set of substitution instances of the formula $\Box p \supset p$. The preservation properties of rules are summarized as follows (Schurz 1997: 52):

Rule	preserves: *truth at a world*	*model-validity*	*frame-validity*
TautR	+	+	+
N, E, RE	–	+	+
Subst	–	–	+

It is also important to prove that certain formulas are not theorems of a logic. This is usually done by giving *semantic counterexamples*. For example, the logic **K** does not contain the theorem T: $\Box A \supset A$, which says that whatever is necessarily true is also true. T seems intuitively to be an indispensable meaning postulate for 'necessity.' A *countermodel* for T is, for example, the Kripke frame F with $W = \{u,v\}$, $R = \{\langle u,v\rangle\}$ (graphically displayed as $u \rightarrow v$), with a valuation function $V(p) = \{v\}$ (it suffices to define V for the variables of the evaluated formula; this is often expressed as a lemma, cf. van Benthem 1983: 26, 2.4). We have $(M,v) \models p$ and thus $(M,u) \models \Box p$, but $(M,u) \not\models p$, and so, $M \not\models T$, whence $F \not\models T$ and thus $T \notin$ **K**.

EXERCISE Give countermodels for $\Box(A \vee B) \supset \Box A \vee \Box B$ (the converse of i) and for $\Diamond A \wedge \Diamond B \supset \Diamond(A \wedge B)$ (converse of ii).

We obtain stronger logics than **K** by imposing structural conditions on frames. The logic **T** = **K** + T is semantically obtained by requiring frames to be *reflexive*, that is to satisfy the frame condition *Ref*: $\forall w$: wRw. We make this more precise by assuming a (first or higher order) *quantificational* language $\mathcal{L}(R)$ which contains the accessibility relation $\underline{R}$ as its only nonlogical predicate and has models of the form $\langle W,R\rangle$. $\models_R$ denotes the standard notion of verification for $\mathcal{L}(R)$-formulas (note that in $\mathcal{L}(R)$-contexts, 'uRv' abbreviates '$\underline{R}$xy'). Then we obtain:

T-CORRESPONDENCE THEOREM For every frame F: $F \models T$ iff $F \models_R$ *Ref*.

PROOF *Right-to-left:* We show that if F is reflexive, then $\Box A \supset A$ is true on every world $w \in W^M$ in every model M based on F. Assume $(M,w) \models \Box A$. Hence $\forall u$: wRu $\Rightarrow$ $(M,u) \models A$. Since F is reflexive, wRw, so $(M,w) \models A$. Hence $(M,w) \models \Box A \supset A$. *Left-to-right:* We show, by contraposition, that if a given $F = \langle W,R\rangle$ is not reflexive, then we can construct a countermodel on F refuting the T-instance $\Box p \supset p$. So assume $w \in W^F$ is an irreflexive point, that is $\neg$wRw. Let p be true at all u with wRu but false at w ($\{u{:}wRu\} \subseteq v(p)$ and $w \notin v(p)$). Call the resulting model M. Now, $(M,w) \models \Box p$, but $(M,w) \not\models p$; so $(M,w) \not\models \Box p \supset p$. Hence $F \not\models T$. Q.E.D.

This is an example of a *correspondence* result. It tells us that the frame condition *Ref* can be *defined* by (or *translated* into) the modal formula T, and *vice versa*. Generally, we say that a modal formula or formula schema $X \in \mathcal{L}$ *corresponds* to a (first or higher order) frame-condition $C_X \in \mathcal{L}(R)$ iff $\forall F \in$ **KF**: $F \models X \Leftrightarrow F \models_R C_X$. In this case, the frames of the modal logic **K** + X (obtained from **K** by adding the axiom schema X) are exactly all frames satisfying C_X. A modal formula (schema) X is called *elementary*, or *first-order definable*, iff X corresponds to a first-order condition C_X. *Correspondence theory* is the field which explores the possibilities of intertranslations between modal and quantificational logic (van Benthem 1983, 1984; our notion of 'correspondence' corresponds to van Benthem's 'global equivalence'; 1983: 48f).

Correspondence results for the five standard principles of alethic PMLs

Modal Principle:	*Frame Condition:*
D: $\neg\Box\bot$	*Ser:* $\forall u\exists v$: uRv (R serial)
T: $\Box A \supset A$	*Ref*: $\forall w$:wRw (R reflexive)
4: $\Box A \supset \Box\Box A$	*Trans:* $\forall$u,v,w: uRv $\wedge$ vRw $\supset$ uRw (R transitive)
B: $\Diamond\Box A \supset A$	*Sym:* $\forall$u,v: uRv $\supset$ vRu (R symmetric)
5: $\Diamond A \supset \Box\Diamond A$	*Euc:* $\forall$u,v,w: wRu $\wedge$ wRv $\supset$ uRv (R euclidean)

PROOF *Exercise* (Chellas 1980: ch. 3.2).

Correspondence results hold only w.r.t. frames; they do *not* say that every *model* validating axiom X satisfies the corresponding frame-condition C_X. It is easy to construct models for a logic **L** which are *not* based on a frame for **L**. We call such models *non-standard* **L**-models. For example, a *irreflexive **T**-model* can be constructed by taking an irreflexive two world frame u $\leftrightarrow$ v where both worlds access each other, and by defining a valuation function V which agrees on both worlds, that is for all p $\in \mathcal{P}$, u $\in$ V(p) iff v $\in$ V(p). It is easily proved for the so obtained model M, by induction on formula-complexity, that (M,u) $\models$ A iff (M,v) $\models$ A holds for all A $\in \mathcal{L}$. Thus, all instances of T are verified on both worlds of M.

The above correspondence results imply that if a PML contains several modal principles, its frames will satisfy all of the corresponding frame conditions. According to the *Lemmon-code* (Lemmon and Scott 1966; see Bull and Segerberg 1984: 20f), we denote normal PMLs as '**KX** . . . ,'' where **X** is a set of additional axiom schemata for these logics (except for special names for logics like **T, S4**, or **S5**). Principle D has been suggested for deontic logics by von Wright. T was suggested by Feys and von Wright for the alethic logic **KT** (von Wright (1951) calls it M). B refers to the 'Browersche system' **KTB** and 4 to Lewis **S4** = **KT4** (both B and 4 have been suggested by Becker), and finally 5 refers to **S5** = **KT5**. Observe the following implication relations between frame-conditions and corresponding logics: (i) *Ref* $\Rightarrow$ *Ser*, thus **KT** = **KDT**; (ii) Sym $\Rightarrow$ (Trans $\Leftrightarrow$ Euc), thus **KB5** = **KB4** = **KB45**; (iii) *Trans* $\Rightarrow$ (*Euc* $\Leftrightarrow$ (*Sym* $\wedge$ *Trans*)), thus **KT5** = **KTB4** = **KDTB45** = **S5**; (iv) *Ser* $\wedge$ *Sym* $\Rightarrow$ *Ref*, thus **KDB** = **KDTB**; (v) *Ser* $\wedge$ *Sym* $\Rightarrow$ (Trans $\Leftrightarrow$ Euc), thus **KDB4** = **KDB5** = **KDB45** = **KDTB45** = **S5**.

PROOF *Exercise* (Chellas 1980: 164).

The possible combinations of these five principles produce 15 *mutually nonequivalent* standard systems of PML (Chellas 1980: 132). Various theorems of PML's stronger than **K** are found in Hughes and Cresswell (1968: ch. 2–4) and Chellas (1980: 131ff); here are some of them.

EXERCISE – PROVE SEMANTICALLY (i) $A \supset \Diamond A$, $\Box^n A \supset A$, $A \supset \Diamond^n A \in$ **KT** ($\Box^n =_{df} \Box \ldots \Box$ n times iterated); (ii) $\Box(\Diamond A \supset B) \supset (A \supset \Box B) \in$ **KB**; (iii) $\Box A \equiv \Box\Box A$, $\Diamond A \equiv \Diamond\Diamond A$, $\Box\Diamond\Box\Diamond A \equiv \Box\Diamond A$, $\Diamond\Box \equiv \Diamond\Box\Diamond\Box A \in$ **S4**; (iv) $\Diamond\Box A \equiv \Box A$, $\Box\Diamond A \equiv \Diamond A \in$ **S5**. – A *modality* m is a possibly empty sequence of $\Box$s and/or $\Diamond$s. Two modalities m_1 and m_2 are **L**-equivalent iff $m_1 p \equiv m_2 p \in$ **L**. The stronger a PML, the more modalities collapse, that is become equivalent. In **S5**, all iterations of modal operators are either equivalent with '$\Diamond$' or with '$\Box$', thus **S5** has only three modalities, namely $\Box$ – Ø – $\Diamond$. For modalities in other systems cf. Chellas (1980: 147ff).

Important semantical operations which *preserve* truth and validity of formulas are the formation of *generated submodels* (*subframes*) and *disjoint unions* of models (frames). This follows from the fact that the truth of $\Box$-formulas in a world w depends only on that part M_w of the given model M which is reachable from w by an R-chain. M_w is called the *w-generated submodel* of $M = \langle W,R,V\rangle$ and is defined as $\langle W_w, R_w, V_w\rangle$, with $W_w = \{u \in W: w\underline{R}u\}$, where $\underline{R}$ is the transitive closure of R, $R_w = R\cap(W_w \times W_w)$, and $V_w(p) = V(p)\cap W_w$ (for all $p \in \mathcal{P}$). The *w-generated subframe* of F is defined accordingly as $F_w = \langle W_w,R_w\rangle$. If **M** is a class of models with pairwise disjoint world sets, then the *disjoint sum* of the models in **M** is defined as $DS(\mathbf{M}) = \langle \cup\{W^M:M \in \mathbf{M}\}, \cup\{R^M:M \in \mathbf{M}\}, \cup\{V^M:M \in \mathbf{M}\}\rangle$; and likewise for DS(**F**). It is straightforward to prove for all formulas A, models M and $w \in W^M$, (i) $(M,w) \models A$ iff $(M_w,w) \models A$, and hence $M \models A$ iff $M_w \models A$, and $F \models A$ iff $F_w \models A$, and (ii) for all model-classes **M** (or fames-classes **F**) with pairwise disjoint world sets, $\mathbf{M} \models A$ iff $DS(\mathbf{M}) \models A$ (or, $\mathbf{F} \models A$ iff $DS(\mathbf{F}) \models A$). A third truth- and validity-preserving operation is the formation of *p-morphic copies* of models and frames. It generalizes the notion of *isomorphic* copy and was introduced by Segerberg (1971: 37; also called 'contraction' by Rautenberg, 'zigzag morphism' by van Benthem and 'reduction' by Chagrov and Zakharyaschev 1997).

PROOF *Exercise* (Hughes and Cresswell 1986: 72f, 80; Chagrov and Zakharyaschev 1997: ch. 2.3).

A model (or frame) which validates the formula set Γ is called a model (or frame) *for* Γ. **M**(Γ), **F**(Γ) denote the set of models, or frames respectively, for Γ. The above results tell us that, for every Γ, the sets **M**(Γ) and **F**(Γ) are closed under the formation of generated submodels (subframes), disjoint unions of frames, and p-morphic models (frames). Preservation results of this sort have various important consequences. A simple example are **S5**-frames. Their accessibility relation is reflexive, symmetric, and transitive and, hence, an *equivalence* relation: it imposes a *partition* onto the world set W into mutually disjoint and exhaustive 'cells' (subsets) $W_1, \ldots, W_n$ (i.e. $W_i \cap W_j = \emptyset$, $\cup_i W_i = W$), such that all worlds in the same cell are mutually accessible, and are inaccessible to worlds in different cells. Hence, each **S5**-frame is the disjoint sum of *universal* frames $\langle W_i, R_i = W_i \times W_i\rangle$. They correspond to Carnap's and Kripke's original **S5**-frames without a relation R. It follows from the generated subframe theorem that all universal frames are in **F**(**S5**); that is **S5** is valid in all universal frames.

A final word on philosophical plausibility. Assume we understand possible worlds as variations of the real world which are possible relative to some *background theory*. If this background theory is logic, then $\Box A$ means that A's truth is determined by principles of logic alone. In this interpretation, all principles of **S5** seem to be valid, in particular all modal iteration principles. For if it is determined by logic that A is true, then it is also determined by logic – namely by *metalogic* – that A's truth is determined by logic: hence $\Box A \supset \Box\Box A$ holds. Likewise, if it is not determined by logic that A is false, then this fact is itself determined by logic: so $\Diamond A \supset \Box\Diamond A$ holds. The same reasoning applies if the background theory contains logic + laws of physics. Then $\Box A$ means that A's truth is determined by logic + laws of physics alone – so, also in this interpretation of $\Box A$, **S5** seems to be the right choice. To avoid confusions: of course, physical necessity is *stronger* than logical necessity, but modal *logics* contain only those principles which are closed under substitution and, hence, are independent from the *content* of a nonlogical symbol. *Proper* physical necessity statements such as 'it is necessary that everything is composed of matter' are content-specific and thus not part of a modal *logic*.

In the above understanding of '$\Box$' we must assume, in order to interpret *iterated* modalities, that either the language of our background theory is *closed* (i.e. it can speak about the truth of its own sentences; cf. ch. VIII of this volume), or that it contains a potentially infinite hierarchy of metalanguages. If we do not make these strong assumptions, then our interpretation will validate only the weaker logic **T**. If we are even more scrupulous and interpret truth-determination as syntactical *derivability* from our background theory, then even **T** will be too strong, as soon as our background theory contains *arithmetic*; the adequate logic for this interpretation is **G**, discussed below. If, on the other hand, '$\Box$' is interpreted in the sense of *tense logic* as 'being true in all future times', then only **S4** but not **S5** is the philosophically adequate logic. These remarks show that even in the narrow realm of *alethic* modal logics, different PMLs are needed for different interpretational purposes.

Axiomatic systems: correctness, completeness, and correspondence

The standard axiomatization of the minimal normal MPL, **K**, is its definition as the smallest set of $\mathcal{L}$-formulas which contains all instances of the axiom schemata Taut and K and is closed under the rules MP and N. The stronger alethic logics **KX**, with **X** $\subseteq$ {D,T,B,4,5}, are axiomatized by adding **X** as the set of so-called *additional* axiom schemata. A formula A is provable in logic **L**, in short $\vdash_{\mathbf{L}} A$, iff A has an **L**-proof, which is a sequence $\langle B_1, \ldots, B_n\rangle$ of formulas such that $B_n = A$, and every B_i $(1 \leq i \leq n)$ is either instance of an axiom schema of **L** or follows from previous members of the sequence by one of the rules of **L**. If $\vdash_{\mathbf{L}} A$, we also call A a theorem of **L**, and identify **L** with the set of its theorems: $\mathbf{L} = \{A: \vdash_{\mathbf{L}} A\}$. A formula A is said to be deducible from formula set Γ, in short $\Gamma \vdash_{\mathbf{L}} A$, iff $\vdash_{\mathbf{L}} \wedge\Gamma_f \supset A$ for some finite subset $\Gamma_f \subseteq \Gamma$. In particular, $\vdash_{\mathbf{L}} A$ iff $\emptyset \vdash_{\mathbf{L}} A$. Finally, Γ is called **L**-consistent iff $\Gamma \nvdash_{\mathbf{L}} p \wedge \neg p$, and **L** is called consistent iff $p \wedge \neg p \notin \mathbf{L}$.

The above axiomatization of PML's is a *Hilbert-style* axiomatization of their modal part together with *an unspecified* (syntactic) determination of tautology-hood. It is rather common for PMLs. If we additionally allow the application of tautological rules TautR, we obtain a *comfortable* way of proving theorems (see exercise below). Of course, various alternative but equivalent axiomatizations are possible. To highlight the relation of **K** to weaker PML's, **K** may equivalently be axiomatized by rules MP + E and axiom schemata Taut + M + C + T (exercise below). $\Gamma \vdash_{\mathbf{L}} A$ may be equivalently defined by the existence of a proof of A from axioms in $\mathbf{L} \cup \Gamma$ with modal rules restricted to formulas which do *not* depend on Γ (see Schurz 1997: 53). There exist also various non-Hilbert-style axiomatizations for PMLs, such as sequent or tableau calculi (see, for example, Fitting 1983; Wansing 1996).

EXERCISE (1.) Prove the theorems of **K**, **T**, **S4** and **S5** listed in § 2.2. Prove that axioms Taut + M + C + T and rules TautR + E are an equivalent axiomatization of **K** (see, for example, Chellas 1980: ch. 4, ch. 8; see also example below).

Example: proof of K from Taut + M + C + TautR + E:

1.	$(\Box(A \supset B) \wedge \Box A) \supset \Box((A \supset B) \wedge A)$	C-instance
2.	$((A \supset B) \wedge A) \equiv (A \wedge B)$	Taut
3.	$\Box((A \supset B) \wedge A) \equiv \Box(A \wedge B)$	E from 2
4.	$(\Box(A \supset B) \wedge \Box A) \supset \Box(A \wedge B)$	TautR from 1,3
5.	$\Box(A \wedge B) \supset \Box B$	M-instance
6.	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	TautR from 4,5

PROOF OF N FROM TAUT + M + C + T + TAUTR + E Assume $\vdash_{\mathbf{L}} A$. Hence $\vdash_{\mathbf{L}} A \equiv \mathsf{T}$, by TautR. So $\vdash_{\mathbf{L}} \Box A \equiv \Box\mathsf{T}$, by E. Because $\Box\mathsf{T}$ is an axiom, we get $\vdash_{\mathbf{L}} \Box A$ by TautR. Q.E.D.

In general, a *normal* PML (i.e. a normal extension of **K**) is defined (after Lemmon and Scott 1966) as any subset $\mathbf{L} \subseteq \mathcal{L}$ which contains **K** and is closed under the rules MP, N, and Subst. Clearly, every normal PML **L** is *representable* as **L** = **KX**, where **X** is *some* set of additional axiom schemata **X** (Schurz 1997: 50, lemma 4). If **X** is recursively enumerable, then **KX** is called (recursively) *axiomatizable* (Chagrov and Zakharyaschev 1997: 495f); if **X** is finite, **KX** is finitely axiomatizable. The class of a normal PMLs forms the infinite lattice Π with **K** as its bottom and $\mathcal{L}$ = the inconsistent logic as its top (for various results on this lattice cf. Chagrov and Zakharyaschev 1997). Not all $\mathbf{L} \in \Pi$ are axiomatizable.

The major properties of axiomatized logics (i.e. axiomatic systems) are their *correctness* and *completeness*. Generally, an axiomatic system L is correct w.r.t. an underlying semantics S iff everything what is L-provable is S-valid, and L is complete w.r.t. S iff everything what is S-valid is L-provable. In modal logics, these notions can be defined w.r.t. models as well as w.r.t. frames, as follows:

1.	**L** is *correct* w.r.t. **F**	iff	$\vdash_{\mathbf{L}} A \Rightarrow \models_{\mathbf{F}} A$ (for all A)
	L is *correct* w.r.t. **M**	iff	$\vdash_{\mathbf{L}} A \Rightarrow \models_{\mathbf{M}} A$ (for all A)
2.	**L** is *w.(eakly) complete* w.r.t. **F**	iff	$\models_{\mathbf{F}} A \Rightarrow \vdash_{\mathbf{L}} A$ (for all A)
	L is *w. complete* w.r.t. **M**	iff	$\models_{\mathbf{M}} A \Rightarrow \vdash_{\mathbf{L}} A$ (for all A)
3.	**L** is *s.(trongly) complete* w.r.t. **F**	iff	$\Gamma \models_{\mathbf{F}} A \Rightarrow \Gamma \vdash_{\mathbf{L}} A$ (for all Γ, A)
	L is *s. complete* w.r.t. **M**	iff	$\Gamma \models_{\mathbf{M}} A \Rightarrow \Gamma \vdash_{\mathbf{L}} A$ (for all Γ, A)
4.	**L** is w./s. *frame-complete* (simpliciter)	iff	**L** is w./s. complete w.r.t. **F**(**L**).
	L is w./s. *model-complete* (simpliciter)	iff	**L** is w./s. complete w.r.t. **M**(**L**).

Completeness *simpliciter* is defined w.r.t. the class of *all* frames or models for a logic. Correctness is the converse property of *weak* completeness. A separate notion of 'strong' correctness ($\Gamma \vdash_{\mathbf{L}} A \Rightarrow \Gamma \models_{\mathbf{M/F}} A$) is not needed: weak correctness implies strong correctness because '$\vdash_{\mathbf{L}}$' is by definition *finitary* ($\Gamma \vdash_{\mathbf{L}} A$ iff $\vdash_{\mathbf{L}} \wedge \Gamma_f \supset A$). Correctness of an axiomatic system **L** is standardly proved as follows: one demonstrates (1) that every axiom of **L** is valid, and (2) that every rule of **L** preserves validity, and concludes, by *induction on the length of the* ***L****-proof of A*, that A is valid w.r.t. the given **M** or **F**. Claim 2 has been established for all normal PMLs, and claim 1 for all PMLs **KX** with $\mathbf{X} \subseteq$ {D,T,B,4,5}. Hence all the latter PMLs are correct w.r.t the corresponding classes of models and frames. Moreover, all PMLs representable as **KX** are correct w.r.t. **M**(**X**) and **F**(**X**).

The standard technique to prove completeness rests on the following *consistency-formulation* of completeness which is classically equivalent:

CONSISTENCY-LEMMA **L** is w. complete w.r.t. **M** [or **F**] iff every **L**-consistent formula A is satisfiable in **M** [or **F**, resp.]; and **L** is s. complete w.r.t. **M** [or **F**] iff every **L**-consistent formula *set* Γ is satisfiable w.r.t. **M** [or **F**, resp.]. *Proof: Exercise.*

Strong completeness is stronger than weak completeness because semantical consequence is not by definition finitary. Strong frame- (or model-) completeness of **L** implies frame- (or model-) *compactness* of **L** in the sense that a formula set Γ is satisfiable on an **L**-frame [or in an **L**-model, resp.] whenever every finite subset of Γ is satisfiable on this **L**-frame [or in that **L**-model, resp.]. Weak completeness plus compactness imply strong completeness. If an axiomatic system **L** is correct and w./s. complete w.r.t. a given class **F** or **M**, then it is said to be w./s. *characterized* by **F** or **M**. This means that the syntactic definition of **L** (and of $\vdash_{\mathbf{L}}$ in the case of s. completeness) coincides with the semantic one.

The canonical technique of proving model-completeness of a normal PML has been introduced by Lemmon and Scott (1966) and Makinson (1966). It is an adaptation of the 'Lindenbaum–Gödel–Henkin' technique to modal logics. It consists in the construction of the so-called *canonical model* $M_c(\mathbf{L})$ of the given logic **L**, which contains maximally consistent formula sets, that is maximal state descriptions, as its worlds (cf. Hughes and Cresswell 1984: 22f; Chellas 1980: 173, def. 5.9):

DEFINITION OF THE CANONICAL MODEL (1) A formula set Γ is maximally **L**-consistent iff Γ is **L**-consistent and no proper extension Δ of Γ is **L**-consistent. (2) The canonical model $M_c(\mathbf{L})$ of **L** (in the given denumerably infinite language $\mathcal{L}$) is defined as $\langle W_c, R_c, V_c \rangle$ where (2.1) W_c is the class of all maximally **L**-consistent formula sets, (2.2) R_c is defined by $\forall u,v \in W_c$: uR_cv iff $\{A: \Box A \in v\} \subseteq u$, and (2.3) for all $p \in \mathcal{P}$, $V_c(p)$ is defined by $\forall w \in W_c$: $w \in V_c(p)$ iff $p \in w$.

It is well-known from truth-functional logic that maximally **L**-consistent formula sets enjoy the following *maximality* properties:

MAXIMALITY-LEMMA For all maximally **L**-consistent sets Δ and formulas A, B: (1) $\Delta \vdash_{\mathbf{L}} A$ implies $A \in \Gamma$ (deductive closure), (2) either $A \in \Delta$, or $\neg A \in \Delta$ (completeness), and (3) $(A \vee B) \in \Delta$ iff ($A \in \Delta$ or $B \in \Delta$) (primeness). Analogous properties exist for $\wedge$ and $\supset$. *Proof: Exercise* (cf. Hughes and Cresswell 1984: 18f).

LINDENBAUM-LEMMA Every **L**-consistent formula set Γ can be extended to a maximally **L**-consistent formula set $\Delta \supseteq \Gamma$ (*proof* cf. Hughes and Cresswell 1984: 19f).

The central idea of the canonical model construction is to prove the following:

TRUTH LEMMA For all $A \in \mathcal{L}$ and $w \in W_c$: $(M_c,w) \models A$ iff $A \in w$.

The three lemmata imply strong completeness of **L** as follows. By maximality lemma 1, **L** is a subset of every world of W_c. So by truth lemma, M_c is an **L**-model. By Lindenbaum-lemma, every given **L**-consistent Γ is subset of some world in W_c. So by truth lemma, Γ satisfied in the **L-**model M_c; Q.E.D. To prove the truth lemma we need the following lemma which guarantees that R_c is well-behaved in the sense that whenever $\Diamond A \in w \in W_c$, then $\exists u$ with wR_cu and $A \in u$:

CANONICAL MODEL LEMMA (1) If $\neg\Box B \in \Gamma$ and Γ is **L**-consistent, then $\{A: \Box A \in \Gamma\} \cup \{\neg B\}$ is **L**-consistent, too. (2) $\forall u \in W_c$: $\Box A \in u$ iff $\forall v$: $uR_cv \Rightarrow A \in v$.

PROOF *Exercise* (Hughes and Cresswell 1984: 21f; Chellas 1980: 172).

PROOF OF THE TRUTH-LEMMA We prove the claim by induction on the complexity of $\mathcal{L}$-formulas. (1) For $A = p \in \mathcal{P}$: $(M_c,w) \models p$ iff $p \in w$ holds for all $w \in W_c$ by definition of V_c. (2) For $A = \neg B$: $(M_c,w) \models \neg B$ iff $(M_c,w) \not\models B$ iff $B \notin w$ by induction hypothesis, iff $\neg B \in w$ by maximality lemma, 2. (3) For $A = B \vee C$: $(M_c,w) \models B \vee C$, iff $(M_c,w) \models B$ or $(M_c,w) \models C$, iff $B \in w$ or $C \in w$ by induction hypothesis and propositional logic, iff $(B \vee C) \in w$ by maximality lemma, 3. (4) For $A = \Box B$: $(M_c,w) \models \Box A$ iff $\forall u$: $wR_cu \Rightarrow (M_c,u) \models A$, iff $\forall u$: $wR_cu \Rightarrow A \in u$ by induction hypothesis and first-order logic, iff $\Box A \in w$ by canonical model lemma, 2. Q.E.D.

The foregoing proofs hold for all normal PMLs and thus establish:

PML-MODEL-CHARACTERIZATION-THEOREM Every normal PML **L** is strongly model-complete, and is strongly characterized by **M**(**L**).

Frame-completeness is stronger than model-completeness: it implies not only that every **L**-consistent formula (set) is satisfiable in *some* **L**-model, but that it is satisfiable in a *standard* **L**-model. So, to prove that **L** is s. frame-complete requires to prove something additional, namely: that the frame of M_c(**L**) is a frame for **L**. Following Fine (1975a), we call normal PMLs satisfying this condition *canonical* (in general, canonicity is relativized to the cardinality of $\mathcal{P}$; but we always assume that $\mathcal{P}$ is denumerably infinite). Canonicity implies strong frame-completeness; whether the reverse direction holds is an open question. Clearly, **K** is canonical because M_c(**K**) is based on a frame. For stronger systems, canonicity has to be proved for each additional axiom schema separately. Axiom schema X is called canonical iff the frame of M_c(**L**) is a frame for **L** whenever **L** contains X. If $X_1, \ldots, X_n$ are canonical, then every **KX** with $\mathbf{X} \subseteq \{X_1, \ldots, X_n\}$ will be canonical, too.

CANONICITY-THEOREM D, T, B, 4 and 5 are canonical.

PROOF *Exercise* (Hughes and Cresswell 1984: ch. 2; Chellas 1980: ch. 5.4).

EXAMPLE *Proof of canonicity of 4:* Assume 4 ∈ **L**. To show that for ∀u,v,w in M_c(**L**), $uR_cv \wedge vR_cw$ implies uR_cw, we assume (a) {A: □A ∈ u} ⊆ v, (b) {A: □A ∈ v} ⊆ w, and prove thereof that (c) {A: □A ∈ u} ⊆ w. Take any □B ∈ u. By deductive closure of canonical worlds, □B ⊃ □□B ∈ u, and thus □□B ∈ u. So □B ∈ v by (a) and B ∈ w by (b).Thus for all □B ∈ u, B ∈ w, which is exactly (c). Q.E.D.

In general, if a normal PML **L** is correct w.r.t. **F**, then it is also correct with respect to every subclass **F′** ⊂ **F**; and if it is w./s. complete w.r.t. **F**, then it is also w./s. complete w.r.t. every superclass **F′** ⊃ **F** (and likewise for models). It often happens that a normal MPL **L** which is characterized by **F**(**L**) is *also* characterized by an interesting subclass **F′** ⊂ **F**(**L**). For example, every canonical **L** ∈ Π is strongly characterized by a single frame, namely the frame of the canonical model M_c(**L**) (this follows direct from the completeness proof). Or, every **L** ∈ Π is w./s. characterized by the class of its generated subframes (which follows from the generated subframe lemma). Another way of producing characteristic subclasses of **F**(**L**) is based on the fact that the following first-order conditions on frames cannot be expressed by modal formulas: *Irr* ¬wRw (irreflexivity), *Asym* uRv ⊃ ¬vRu (asymmetry), *Antisym* uRv ∧ u ≠ v ⊃ ¬vRu (antisymmetry), *Intrans* uRv ∧ vRw ⊃ ¬uRw, and *Anticon* ∀u,v,w: u ≠ v ≠ w ∧ uRw ⊃ ¬vRw (anticonvergence). In other words, *correspondence* fails for these conditions in the right-to-left direction. For **K**, this can be proved by the technique of *unraveling*, which is a validity-preserving transformation of arbitrary models into irreflexive, asymmetric, and intransitive models (due to Dummett and Lemmon 1959; see Bull and Segerberg 1984: 45). Thus, **K** is also strongly characterized by all irreflexive, asymmetric, and intransitive frames.

Characterization by subclasses is important for the PML's of *ordering relations*. The technique of *bulldozing* introduced by Segerberg (1971: 78ff) transforms every reflexive and transitive model M into a validity-preserving partially ordered model M; if M is merely transitive then the ordering of M's frame is strict. It follows from this that **K4** is strongly characterized by the class of strict partially ordered frames, and **S4** by the class of partially ordered frames. Finally, *ramification* transforms arbitrary [reflexive, transitive] models into validity-preserving [reflexive, transitive, resp.] models based on *tree-frames* (Chagrov and Zakharyaschev 1997: 32–5). Tree models represent branchings of possible future states in time and are important for the logic of causality and agentship (cf. Kutschera 1993, Prendinger and Schurz 1996).

A brief remark on *classical* modal logics concludes this section. They are weaker than **K** and are mainly used for nonalethic (e.g. epistemic, deontic) interpretations of the modal operator (for details see Segerberg 1971: ch. 1 and Chellas 1980: part III). The minimal classical modal logic, **E**, is axiomatized by Taut, MP, and the rule E. **E** allows it to regard the intensional operator □ as applying to *propositions*; this requires truth-preservation of □ under replacements of logically equivalent sentences. **M** = **E** + M is the minimal *monotonic* and **C** = **E** + M + C the minimal *regular* modal logic. If we finally add ⊤ we get an alternative axiomatization of **K**. Semantically, classical modal logics are characterized by so-called *neighborhood* frames, which are pairs $\langle W,N \rangle$ with $W \neq \phi$ a possible world set and $N:W \rightarrow Pow(Pow(W))$ a function assigning to each world $w \in W$ a neighborhood $N(w)$ which contains exactly those 'propositions' (i.e. W-subsets) which are necessarily true at w. Completeness proofs proceed via canonical neighborhood models: **E** is characterized by the class of all neighborhood frames, and the semantic conditions corresponding to **M**, **C**, and **T** are closure of neighborhoods under supersets, finite intersections, and containment of W. **K**-neighborhood-frames with all three properties can be transformed into point-wise equivalent Kripke frames, and **C**-neighborhood frames can be transformed into Kripke-frames with an additional set of so-called 'queer' worlds (Segerberg 1971: 23ff).

Decidability and finite model property

A logic $\mathbf{L} \subseteq \mathcal{L}$ is called *decidable* iff for every $A \in \mathcal{L}$ it can be decided after a finite number of primitive computation steps whether or not $A \in \mathbf{L}$. Of course, the mere axiomatizability (i.e. recursive enumerability) of a logic does *not* imply its decidability. It is a famous fact that a logic **L** is decidable iff the theorems of **L** as well as the non-theorems of **L** (i.e. the elements of $\mathcal{L} - \mathbf{L}$) are recursively enumerable (Chagrov and Zakharyaschev 1997: 492). A logic **L** is said to have the *finite model property* (f.m.p.) iff every **L**-consistent formula A is satisfiable on a *finite* **L**-model. (Likewise for the 'finite frame-property.') A standard way of proving the decidability of an axiomatizable logic is by proving that it has the f.m.p. For, we can effectively enumerate all finite models of a given formula A and test whether they are A-countermodels. So if an axiomatizable logic **L** has the f.m.p., then after a finite number of steps either the enumeration of **L**-theorems will output A, or the enumeration of $\mathcal{L}$'s finite models will produce an A-countermodel (Chagrov and Zakharyaschev 1997: 492). Note, however, that there are also decidable logics which do *not* have the f.m.p. (Gabbay 1976: 258–65).

Note the following fundamental *f.m.p.-theorem:* For all $\mathbf{L} \in \Pi$: **L** has the f.m.p. $\Leftrightarrow$ **L** has the finite frame property $\Leftrightarrow$ **L** is w. complete w.r.t. **L**'s finite frames. The second equivalence is an immediate consequence of the first, which has been proved by Segerberg (1971: 29ff) as follows: (1) for every model there exists an elementary equivalent *distinguishable* model (where no two worlds verify the same formulas); and (2) if a finite distinguishable models validates **L**, then its frame is an **L**-frame.

A standard technique to produce finite models for a given formula or finite formula set is *filtration*. Assume Γ is a set of formulas closed under subformulas (i.e. if $A \in \Gamma$, and B is a subformula of A, then $B \in \Gamma$). Given $M = \langle W,R,V \rangle$, two worlds $u, v \in W^M$ are called Γ-equivalent, in short $u \equiv_\Gamma v$, iff they verify the same formulas in Γ (i.e. iff $\forall A \in \Gamma$: $(M,u) \models A \Leftrightarrow (M,v) \models A$). For $w \in W^M$, $[w]_\Gamma =_{df} \{u \in W^M: w \equiv_\Gamma u\}$ denotes the Γ-equivalence class of w. Then, a model $M_\Gamma = \langle W_\Gamma,R_\Gamma,V_\Gamma \rangle$ is called a *Γ-filtration of M*, iff $M_\Gamma = \{[w]_\Gamma: w \in W^M\}$, $V_\Gamma(p) = \{[w]_\Gamma: w \in v(p)\}$ for all $p \in \mathcal{P}$, and R satisfies two conditions ($u, v \in W^M$): (F1): If uRv, then $[u]_\Gamma\ R_\Gamma\ [v]_\Gamma$, and (F2): If $[u]_\Gamma\ R_\Gamma\ [v]_\Gamma$, then $\forall A \in \Gamma$: $(M,u) \models \Box A \Rightarrow (M,v) \models A$. Note that there exist several Γ-filtrations of a given model M. The frame $\langle W_\Gamma,R_\Gamma \rangle$ is the corresponding Γ-filtration of $\langle W,R \rangle$.

FILTRATION THEOREM If M_Γ is a Γ-filtration of M, then $\forall A \in \Gamma\ \forall w \in A^M$: $(M,w) \models A$ iff $(M_\Gamma, [w]_\Gamma) \models A$. *Proof: Exercise* (Hughes and Cresswell 1984: 139).

M_Γ is a finite model whenever Γ is finite. Thus, by filtering a model M for an **L**-consistent formula A through the (finite) set *subf(A)* of A's subformulas, we obtain a finite model $M_{subf(A)}$ verifying A. To prove by this method that **L** has the f.m.p. requires in addition to prove that the filtered model is indeed an **L**-model. A simple way to do this is to show that the filtered frame $\langle W_{subf(A)}, R_{subf(A)} \rangle$ is a frame for **L**. This is easy for the logics **K**, **KD**, and **KT**, since one can prove that every filtration of a frame preserves seriality or reflexivity (Chellas 1980: 105). For other standard systems such as **KB**, **K4**, **S4**, etc., special filtrations are necessary to demonstrate preservation of the corresponding frame-properties (Chellas 1980: 106ff). Segerberg has proved that all normal extensions of **K45** have the f.m.p. and, thus, can be classified as the logics of certain simple frame classes (Segerberg 1971: 123ff).

We finally remark that, though f.m.p. proves decidability for axiomatizable logics, it does not produce a practically feasible decision method. Practically feasible decision methods for standard systems are, for example, *tableau methods* (Hughes and Cresswell 1968: ch. 5–6; Chagrov and Zakharyaschev 1997: ch. 3.4).

More metalogical results on PMLs

Further examples of axiom schemata which are both first-order definable and canonical are:

Axiom schema:	*Corresponding first-order condition:*
($G^{k,l,m,n}$) $\Diamond^k\Box^l A \supset \Box^m\Diamond^n A$	R is k, l, m, n-incestual: $\forall u,v,w,w'$: $uR^k v \wedge vR^m w \supset \exists w'(vR^l w' \wedge wR^n w')$
0.3: $\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	R is locally strongly connected: $\forall u,v$: $\exists w(wRu \wedge wRv) \supset (uRv \vee vRu)$
0.3*: $\Box(A \wedge \Box A \supset B) \vee \Box(B \wedge \Box B \supset A)$	R is locally connected: $\forall u,v$: $\exists w(wRu \wedge wRv) \wedge u \neq v \supset (uRv \vee vRu)$
0.2: $\Diamond\Box A \supset \Box\Diamond A$	R is locally strongly convergent: $\forall u,v$: $\exists w(wRu \wedge wRv) \supset \exists w'(uRw' \wedge vRw')$
0.2*: $\Diamond(A \wedge \Box B) \supset \Box(A \vee \Diamond B)$	R is locally convergent: $\forall u,v$: $\exists w(wRu \wedge wRv) \wedge u \neq v \supset \exists w'(uRw' \wedge vRw')$
Dense: $\Box\Box A \supset \Box A$	R is dense: $\forall u,v$: $Ruv \supset \exists w(uRw \wedge wRv)$
Triv: $\Box A \equiv A$	Every world reaches only itself: $\forall u,v$: $uRv \supset u = v$
Ver: $\Box\bot$	Every world is a dead end: $\forall u,v$: $\neg uRv$
Alt_n: $\Box A_1 \vee \Box(A_1 \supset A_2) \vee \ldots \vee \Box(A_1 \wedge \ldots \wedge A_n \supset A_{n+1})$	Every world reaches at most n distinct worlds: $\forall u, v_1, \ldots, v_{n+1}$: $\wedge \{uRv_i: 1 \leq i \leq n\} \supset \vee \{v_i = v_j: 1 \leq i < j \leq n\}$

$G^{k,l,m,n}$ is a very general schema introduced by Lemmon and Scott (1966) (Hughes and Cresswell 1984: 42); note that D is **K**-equivalent with $G^{0,1,0,1} = \Box A \supset \Diamond A$, T = $G^{0,1,0,0}$, B = $G^{0,0,1,1}$, 4 = $G^{0,1,2,0}$, 5 = $G^{1,0,1,1}$, 0.2 = $G^{1,1,1,1}$, Dense = $G^{0,2,1,0}$. **S4.2** = **S4** + 0.2. Sahlqvist (1975) has proved first-order definability and canonicity for a class of axiom schemata which is even more general than $G^{k,l,m,n}$ (cf. Chagrov and Zakharyaschev 1997: ch. 10.3). The schemata 0.3, 0.3* (introduced by Lemmon) and 0.2, 0.2* (introduced by Geach) are important for the modal logics of orderings; 0.3, 0.3* are equivalent for reflexive frames, and 0.2, 0.2* for serial frames. **S4.3** = **S4** + 0.3 is the logic of linear orderings. Likewise, **K4.3** is the logic of strict linear orderings and **KD4.3** the logic of strict linear orderings without last element. By adding Dense one obtains the logics of corresponding dense orderings. Ver and Triv are famous because they are characterized by the two singleton frames $\langle\{w\}, \langle w,w\rangle\rangle$ and $\langle\{w\}, \emptyset\rangle$, respectively, and every consistent $\mathbf{L} \in \mathbf{\Pi 0}$ is either contained in Triv or in Ver (Makinson's theorem). The logics **S5**(**Alt**$_\mathbf{n}$) are the only consistent extensions of **S5** (Scroggs' theorem; **S5Alt**$_\mathbf{1}$ **= KTriv**). For more details and canonical axioms see, for example, Segerberg (1971); Hughes and Cresswell (1984); or Chagrov and Zakharyaschev (1997).

Let us turn to examples where completeness and/or correspondence fails. We call an axiom schema X (and the corresponding logic **KX**) *non-compact* iff it is weakly but not strongly frame-complete, and *frame-incomplete* iff it is not even weakly frame-complete. Examples of axiom schemata which are both non-compact and not elementary (not first-order definable) are Löb's axiom G (also called W) and McKinsey's axiom 0.1, along with their corresponding frame-condition:

(G)	$\Box(\Box A \supset A) \supset \Box A$	R is transitive and terminal, that is there are no infinite R-chains $w_1Rw_2 \ldots w_nRw_{n+1} \ldots$
(0.1)	$\Box\Diamond A \supset \Diamond\Box A$	For no $w \in W$ there exist disjoint nonempty W-subsets U, V, such that all $w \in \{w^*{:}wRw^*\}$ have R-successors in U and V

G-frames are irreflexive, since a reflexive w implies the infinite chain wRwRw . . . As a result, **KG** contains 4, but neither T nor D (Hughes and Cresswell 1984: 101). That the G-corresponding condition C_G at the right side (proof see van Benthem 1984: 195f) is genuinely second-order is seen as follows. Consider the infinite formula set $\Delta = \{C_G\} \cup \{x_iRx_{i+1}: i \in \omega\}$. Every finite subset of Δ is satisfiable in a G-frame. Since first-order logic is compact, it follows that if C_G were first-order, then Δ would be satisfiable in a G-frame. But it is not, since by asymmetry of R this would imply an infinite ascending R-chain. So C_G is not first-order (Chagrov and Zakharyaschev 1997: 166). That **KG** is not canonical can be proved by showing that the frame of M_c(**KG**) contains reflexive worlds and, thus, is not a **KG**-frame: this follows from the fact that the so-called Solovay's logic **S** = **KG** + T is consistent, and hence, produces reflexive worlds in M_c(**KG**) (Chagrov and Zakharyaschev 1997: 165). *Weak* completeness of **KG** is proved by a suitable filtration of M_c(**KG**) (Hughes and Cresswell 1986: 47ff).

Correspondence for the second-order condition corresponding to McKinsey's axiom 0.1 is established in Fine (1975b). **K0.1** has the f.m.p. and thus is w. frame-complete (Fine 1975a), but it is not canonical (Goldblatt 1991). The logic **KG** ('G' for 'Gödel') has become famous because it allows a translation of Gödel's incompleteness proof for first-order arithmetics into modal logic. If one translates $\Box A$ into the arithmetical language with Gödel-numbering g and provability predicate Pr(x), by the translation function $t(\Box A) = Pr(g(t(A)))$, then **KG** contain all modal theorems which are valid in this arithmetical interpretation, and Gödel's incompleteness results have a direct translation into the modal language (for details see Smorynski 1984). Also McKinsey's axiom is remarkable, for two reasons. First, 0.1 becomes canonical if it is added to **K4** or **S4**:

K4.1 = **K4** + 0.1 and **S4.1** = **S4** + **0.1** are first-order definable and canonical (proved in Lemmon and Scott 1966: 75). **K4.1**-frames are defined by the transitivity of R and the condition that every world reaches a 'dead end' ($\forall u\exists v$: uRv $\wedge$ $\forall w$(vRw $\supset$ v = w)); and **S4.1**-frames are additionally reflexive (van Benthem 1984: 202). Second, **S4.1** is then a simple example of a canonical PML with a frame-incomplete quantificational counterpart (§3.1 below).

All **K45**-extensions and all **S4.3**-extensions are weakly frame-complete. Lemmon conjectured in 1966 that all normal PMLs are w. frame-complete. In 1974, Fine and Thomason gave first examples of *frame-incomplete* PMLs. A standard way of proving the *frame-incompleteness* of **L** $\in$ Π is the following: prove (i) that **F**(**L**) validates a certain formula schema X, and (ii) that X is not derivable in **L**, by specifying a class **M** of *non-standard* models of **L** which falsifies X. (Note that **M** cannot be standard because then **M** would also verify X.) By *correctness* w.r.t. models, this implies that $\nvdash_{\mathbf{L}}$ X, and hence, that **L** is frame-incomplete.

Model-classes for a given **L**, even if nonstandard, must preserve validity under the rule of substitution. Such model-classes have been introduced as so-called *general frames* by Thomason (1972) (for details see Chagrov and Zakharyaschev 1997: ch. 8). A *general frame* G is defined as a pair G = ⟨F,Prop⟩ where F is an ordinary frame and Prop $\subseteq$ Pow(W) is a set of 'valuation-admissible' subsets of W which is closed under intersection, relative complement, and under the operation '$\Box$:W $\to$ W' defined by $\Box$X = {w: $\forall$u(wRu $\supset$ u $\in$ X}. The class **M**(G) of G-models is the class of all models based on F with valuation function V:$\mathcal{P}$ $\to$ Prop taking values in Prop. By definition, G $\models$ A iff **M**(G) $\models$ A. This definition entails, by the closure conditions on Prop, that whenever a general frame G validates A, then G validates every substitution instance s(A) of A. Moreover, to every model M = ⟨W,R,V⟩ there corresponds a *minimal* general frame G_M defined as ⟨⟨W,R⟩,$Prop_M$⟩ with $Prop_M$ = the set of W-subsets which are the value of some $\mathcal{L}$-formula under V (Chagrov and Zakharyaschev 1997: 237). It follows that every substitution-closed formula set and in particular every logic **L** $\in$ Π which is valid in M must also be valid in G_M. As a result, model-completeness of a logic implies its completeness w.r.t. general frames. (For details on general frames and their connection to *modal algebras* see Chagrov and Zakharyaschev 1997.)

A simple example of a frame-incomplete PML is van Benthem's logic **KVB**, where VB = $\Diamond\Box\bot \vee \Box(\Box(\Box B \supset B) \supset B)$. It is easily checked that every frame for VB satisfies the first-order condition that every world is a dead end or reaches a dead end, and hence, validates the axiom $\Diamond$Ver = $\Diamond\Box\bot \vee \Box\bot$. Van Benthem constructs a countably infinite general frame with allowable values based on finite and cofinite W-subsets which validates VB but falsifies $\Diamond$Ver (Hughes and Cresswell 1984: 57ff). Van Benthem's examples shows also that first-order definability does not imply frame-completeness of a logic (which was an earlier conjecture). That also the reverse implication relation does not hold was demonstrated by an example of a canonical logic which is not first-order definable, given by Fine (1975a), namely the logic **KF** = **K** + F $=_{df}$ $\Diamond\Box A \supset \Diamond\Box(A \wedge B) \vee \Diamond\Box(A \wedge \neg B)$.

Both examples show that there is no simple relationship between completeness and correspondence. First of all, correspondence has *two sides*. Modal formulas may or may not have corresponding frame-conditions (correspondence I), and frame-conditions may or may not have corresponding modal formulas (correspondence II) (van Benthem 1984: 192, 211). Concerning correspondence I, van Benthem (1984: 169ff) shows that every modal formula has a corresponding second-order frame condition; so the only interesting question is whether a modal formula is elementary. Concerning correspondence II, various examples of modally undefinable first-order conditions have been given in §2.3. General theorems about correspondence I and II are found in van Benthem (1983, 1984). The following connections between frame-completeness and correspondence I have been proved in Fine (1975a): (1) If $\mathbf{L} \in \Pi$ is first-order definable and w. frame-complete, then **L** is canonical, and (2) If $\mathbf{L} \in \Pi$ is natural, then **L** is first-order definable. Naturality, as defined by Fine, is a stronger property than canonicity (a generalization of Fine's theorem for general frames in terms of 'D-persistence' is given by van Benthem (1983); see Chagrov and Zakharyaschev 1997: 341–4).

For many purposes, one needs PMLs with several different modal operators, for example an alethic-deontic PML for the is–ought problem (Schurz 1997). A *multimodal language* $\mathcal{L}_I$ contains a set $\{\Box_i: i \in I\}$ of modal operators (I an index set). The simplest kind of a normal PML in $\mathcal{L}_I$ is a so-called *combination* (or join) of normal monomodal $\Box_i$-logics $\{\mathbf{L}_i: i \in I\}$, denoted by $\otimes\{\mathbf{L}_i: i \in I\}$, and defined as the smallest normal PML in $\mathcal{L}_I$ containing every $\mathbf{L}_i$. Frames for these logics have the form $\langle W, \{R_i: i \in I\}\rangle$. Syntactically, $\otimes\{\mathbf{L}_i: i \in I\}$ is obtained from the $\mathbf{L}_i$ $(i \in I)$ by joining their representative axiom sets $\mathbf{X}_i$ and under substitution in the combined language $\mathcal{L}_I$. Hence, $\otimes\{\mathbf{L}_i: i \in I\}$ is representable as $\mathbf{K_I X_I}$ with $\mathbf{X}_I = \cup_i \mathbf{X_i}$. Instead of proving metalogical properties like completeness, etc. for all possible multimodal combinations separately, it is more desirable to prove general *transfer theorems* in the following sense: whenever all $\mathbf{L}_i$ have a certain property, then $\otimes\{\mathbf{L}_i: i \in I\}$ has it, too. The following general transfer theorem holds for combined multimodal logics: (1) Weak and strong frame-completeness, canonicity and f.m.p. transfer from the $\mathbf{L}_i$ $(i \in I)$ to $\otimes\{\mathbf{L}_i: i \in I\}$; and (2) decidability, interpolation, and Halldén-completeness transfer under presupposition of weak completeness of the $\mathbf{L}_i$ $(i \in I)$. The theorem was independently proven by Kracht and Wolter (1991) and Fine and Schurz (1996) (the latter paper was been written up in 1990 but its publication was delayed). If a multimodal logic contains in addition *interactive* axioms which relate distinct modalities (e.g. $\Box_1\Box_2 A \supset \Box_2\Box_1 A$), then transfer theorems are possible only in special cases (Fine and Schurz 1996: 210ff, for such examples). The investigation of combined logics and transfer has become a topic of increasing interest in modal logics; for a survey see Kracht and Wolter (1997).

3 Modal Quantificational Logics (QMLs)

Fixed domain and rigid designators: Q1MLs

$\mathcal{LQ}1$ is the object language modal quantificational logics of 'type 1,' in short: Q1MLs. It contains a set $\mathcal{V}$ of individual variables (x, y, . . .), in short: variables, a set C of individual constants (a, b, . . .), in short: constants, and for each $n \geq 0$, a set $\mathcal{R}^n$ of n-ary relation symbols (F, G, Q, . . .). All these sets are denumerably infinite (0-ary relation symbols are propositional variables). For reasons of simplicity we omit function symbols; thus singular terms, denoted by t, t_1, t_2, . . . , are constants or variables; $\mathcal{J}$(the

set of all terms) $=_{df}$ $\mathcal{V} \cup C$. The new primitive logical symbols are the universal quantifier $\forall$ ($\forall$x = 'for all x:'), and the identity symbol, =. The existential quantifier $\exists$ ($\exists$x = 'there exists an x:') is defined as usual by $\exists xA =_{df} \neg\forall x\neg A$. $\mathcal{LQ}1$ is again identified with the set of its (well-formed) formulas, which are recursively defined by the following clauses: (1) $t_1, \ldots, t_n \in \mathcal{J}$, $Q \in \mathcal{R}^n \Rightarrow Qt_1 \ldots t_n \in \mathcal{LQ}1$; (2) A, B $\in \mathcal{LQ}1 \Rightarrow \neg A$, $A \vee B$, $\Box A \in \mathcal{LQ}1$; and (3) $x \in \mathcal{V}$, $A \in \mathcal{LQ} \Rightarrow \forall xA \in \mathcal{LQ}1$.

We assume acquaintance with the notions of bound and free *occurrences* of variables. Variable x is called free in A iff A contains at least one free x-occurrence. $\mathcal{V}_f(A)$ = the set of free variables in A; likewise for $\mathcal{V}(A)$, $C(A)$, $\mathcal{R}^n(A)$. A* is an *alphabetic variant* of formula A iff A* results from A by replacing every bound occurrence of some variables $x_1, \ldots x_n$ in A by variables $y_1, \ldots y_n$, respectively, provided (for each $1 \leq i \leq n$) that no x_i-occurrence in A lies in the scope of an $\forall y_i$-quantifier and no free y_i-occurrence in A lies in the scope of an $\forall x_i$-quantifier. A[t/x] denotes the result of the correct substitution of term t for variable x in A and is defined as the result of the replacement of every free occurrence of x in A* by t; where A* is the first alphabetic variant of A (according to a given formula enumeration) in which x does not occur in the scope of a quantifier binding t. $A[t_{1-n}/x_{1-n}]$ denotes the result of the correct (simultaneous) substitution of t_i for x_i in A (for all pairwise distinct x_i).

The notion of a *frame* remains the same for all kinds of QML-semantics. The simplest way of extending Kripke models to modal quantificational languages are *Q1-models*. They contain one fixed domain D of objects, which is the same for all worlds in W, and assume that singular terms are rigid – so only the interpretation of the relation symbols is world-relative. More precisely, a Q1-model is a quadruple $M = \langle W,R,D,V \rangle$ where $\langle W,R \rangle$ is a frame, $D \neq \emptyset$ is a nonempty domain of individuals, and the valuation function V is defined as follows: (1) $V: \mathcal{J} \rightarrow D$, hence $\forall t \in \mathcal{J}$. $V(t) \in D$, and (2) for all $n \geq 0$, $V: W \times \mathcal{R}^n \rightarrow D^n$, hence $V(w,R) =_{df} V_w(R) \subseteq D^n$. The value $V_w(R)$ is also called the 'extension' of R at world w, and the partial function $V(R){:}W \rightarrow D^n$ is called R's *intension* (this view of intension' goes back to Carnap). The restriction of V to constants and relation symbols is often called an *interpretation* of $\mathcal{LQ}1$, and the restriction of v to variables an *assignment* for variables. Because we treat free variables and constants semantically on a par, we don't need to distinguish between closed and open formulas. This setting is close to Machover (1996: 151f). Of course, variations are possible. For example, one may drop constants and let free variables play their role, as in Hughes and Cresswell (1984); or one may give free variables the closure-interpretation, as in Fine (1978).

M[x:d] denotes a model which is like M except that v^M assigns $d \in D$ to x; and similar for $M[x_{1-n}/d_{1-n}]$. The definition of '$(M,w)\models A$' is as follows: for atomic formulas, $(M,w)\models Rt_1 \ldots t_n$ iff $\langle V(t_1), \ldots, V(t_n)\rangle \in v_w(R)$ and $(M,w)\models t_1 = t_2$ iff $V(t_1) = V(t_2)$; for $A = \neg B$, $B \vee C$, $\Box B$ as in the propositional case; and for quantified formulas: $(M,w) \models \forall xA$ iff $\forall d \in D$, $(M[x{:}d],w) \models A$. The other semantical notions are as in the propositional case. The *coincidence lemma* tells us that $(M,w) \models A[t/x]$ iff $(M[x{:}v^M(t)],w) \models A$ (*Proof: exercise* (Hughes and Cresswell 1984: 168).

DEFINITION OF NORMAL Q1MLS Given a PML **KX**, its *Q1-counterpart* is denoted as **Q1KX** and is defined as the smallest set of $\mathcal{LQ}1$-formulas which contains all $\mathcal{LQ}1$-instances of the axiom schemata of **KX** plus the following axiom schemata for quantification (UI, $\forall$, UG, BF) and identity (I, rISub, rI¬) (for all $x \in \mathcal{V}$, $t \in \mathcal{J}$):

UI: $\forall xA \supset A[t/x]$ ('universal instantiation')
BF: $\forall x\Box A \supset \Box\forall xA$
$\forall 1$: $\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$
$\forall 2$: $A \supset \forall xA$, provided x is not free in A.
I: $t = t$
rISub: $t_1 = t_2 \supset (A[t_1/x] \supset A[t_2/x])$ ('rigid identity-substitution')
rI¬: $\neg t_1 = t_2 \rightarrow \Box\neg t_1 = t_2$ ('rigid identity w.r.t. ¬)

and which is closed under the rules of **KX** (TautR, MP, N) and under the rule:

UG: $A/\forall xA$ ('rule of universal generalization').

Provability $\vdash_{\mathbf{L}} A$ and deducibility $\Gamma \vdash_{\mathbf{L}} A$ is defined as for PMLs.

RECAPITULATION OF NONMODAL QL Prove the dual axiom of 'existential instantiation' EI: $A[t/x] \supset \exists xA$, and the dual rule of 'existential generalization' EG: $A \supset B/\exists xA \supset B$, provided $x \notin \mathcal{V}_f(B)$. Prove the equivalence of UG with UGt: $A[x/t]/\forall xA$, provided $t \notin \mathcal{J}(A)$. Prove that our axiomatization UI + ∀1 + ∀2 + UG (also used by Fine 1978) is equivalent with UI + UG*: '$A \supset B/A \supset \forall xB$, provided $x \notin \mathcal{V}_f(A)$' (used, e.g. in Hughes and Cresswell 1984: 166).

SYNTACTICAL THEOREMS ABOUT Q1MLS (1) The rule UG is neither valid nor model-admissible, but merely frame-admissible. (2) BF is valid in every Q1-model. (3) The converse Barcan formula, cBF: $\Box\forall xA \supset \forall x\Box A$, is a **Q1K**-theorem. (4) The rigidity principle rI: $t_1 = t_2 \supset \Box(t_1 = t_2)$ is a **Q1K**-theorem, and the rigidity axiom rI¬ is a **Q1B**-theorem. (5) The formula $\Box\exists xA \supset \exists x\Box A$ is invalid.

PROOF *Exercise* (see examples below; for 3 see Hughes and Cresswell 1968: 143; for 4 see Schurz 1997: fn.s 108, 109). The counterintuitivity of 5 was illustrated by Quine (1953, p. 148) as follows: it necessary that one player will win, but for no one of the players is it necessary that just he will win (cf. Hughes and Cresswell 1968, p. 197).

DISPROOF OF MODEL-ADMISSIBILITY OF UG By the following countermodel $M = \langle\{w\}, \emptyset, \{d_1, d_2\}, V\rangle$ with $V_w(F) = \{d_1\}$. It yields $M \models Fa$ but $M \not\models \forall xFx$.

PROOF OF FRAME-VALIDITY OF UG BY CONTRAPOSITION Assume $\langle W,R\rangle \not\models \forall xFx$. So there exist D, V, w such that for $M = \langle W,R,D,V\rangle$ and $w \in W^M$, $(M,w) \not\models \forall xFx$. Hence $\exists d \in D^M$: $(M[x{:}d],w) \not\models Fx$. Since the model $M[x{:}d]$ is based on $\langle W,R\rangle$, this implies that $\langle W,R\rangle \not\models Fx$. Q.E.D.

A general definition of normal Q1MLs requires a suitable formulation of the *rule of substitution for predicates*. This rule was first described by Kleene (1971: 155–62) and is explained as follows. A substitution instance of formula A w.r.t. an n-ary predicate Q in 'name form variables' $z_1 \ldots z_n$ is a formula A* which results from the simultaneous replacement of every occurrence of a term-instance $Qt_1 \ldots t_n$ in A** by a corresponding term-instance $B[t_{1-n}/z_{1-n}]$, for a given formula ('complex predicate') B; where A** is the first alphabetic variant of A in which no free variable of B other than

$z_1, \ldots, z_n$ gets bound (for details see Schurz 1995: 45–52). Important in our context is the following *QML-substitution-theorem:* frame-validity of Q1ML-formulas is preserved under substitution for predicates (proof see Schurz 1997: 46–8). This theorem guarantees that for every frame class **F**, **L**(**F**) will be closed under substitution and, hence, will be a *normal Q1-logic*. Moreover, our notion of substitution allows us to define *a normal* **Q1**-logic as any formula set **L** $\subseteq$ $\mathcal{LQ}1$ which contains **Q1K** and is closed under the rules of **Q1K** and under substitution for predicates. **Q1**Π denotes the lattice of normal Q1-logics.

As in the propositional case, every **L** ∈ **Q1**Π is *representable* (but not necessarily axiomatizable) as **Q1KX**. **L** ∈ **Q1**Π is called *propositionally representable* iff **L** = **Q1KX** for some **X** consisting solely of *propositional* axiom schemata – in other words, iff **L** is the Q1-counterpart of the PML **KX**. Propositionally representable Q1-logics are the standard case. However, **X** may also contain additional quantificational (or identity) axiom schemata – on two reasons: First, some frame-complete PMLs have frame incomplete Q1-counterparts, which need additional $\mathcal{LQ}1$-axioms to become frame-complete (cf. **Q1S4.1** below). Second, there exist interesting cases of additional schemata which are only characterizable by nonstandard model-classes, such as Fine's anti-Haecceitistic axiom **H**.

Correspondence, correctness, and w./s. completeness w.r.t. models or frames (and related notions) are defined as in the propositional case. Of course, Q1-logics do neither have the f.m.p. w.r.t the domain, nor are they decidable, because nonmodal first-order logic lacks these properties. Correctness of Q1-logics is proved, as usual, by showing that all **Q1L**-axioms are valid on all frames, and that all **Q1K**-rules preserve frame-validity; this was done above. The correctness-proof also establishes that every propositionally representable Q1-logic corresponds to the same class of frames as its propositional counterpart. This gives us a following *frame transfer theorem* from PMLs to their Q1-counterparts: For every PML **KX: F**(**KX**) = **F**(**Q1KX**). Hence, if a frame-condition C_X corresponds to **KX**, then it corresponds also to **Q1KX**.

As in non-modal QL, the domain of the *canonical model* of a Q1ML is constructed from the =-equivalence classes of terms. On this reason, the canonical worlds need not only be maximally **L**-consistent formula sets, they also have to be 'ω-complete'. The canonical model $\mathbf{M}_c(\mathbf{L},\Delta)$ of a Q1-logic **L** is explicitly relativized to a saturated formula set Δ which extends the given **L**-consistent formula set Γ and *fixes the rigid term identities*. Implicitly, the notion of ω-completeness and the canonical model is also relativized to the *term set* $\mathcal{J}(\mathcal{LQ}1)$ of the given denumerably infinite language $\mathcal{LQ}1$.

DEFINITION OF CANONICAL Q1-MODELS (1) A formula set $\Gamma \subseteq \mathcal{LQ}1$ is ω-*complete* (w.r.t. $\mathcal{LQ}1$) iff for every $A \in \mathcal{LQ}1$: $\Gamma \vdash_{\mathbf{L}} \forall x A$ iff $\Gamma \vdash_{\mathbf{L}} A[t/x]$ for every $t \in \mathcal{J}(\mathcal{LQ}1)$; Γ is called ***L**-saturated* iff it is both maximally **L**-consistent and ω–complete. (2) The canonical model $\mathbf{M}_c(\mathbf{L},\Delta) = \langle W_c, R_c, D_c, V_c \rangle$ of **L** ∈ **Q1**Π for the **L**-saturated formula set Δ in given language $\mathcal{LQ}1$ is defined as follows: (2.1) W is the set of all **L**-saturated $\mathcal{LQ}1$-formula sets w which preserve the Δ-identities; that is for all t_1, t_2: $t_1 = t_2 \in w$ iff $t_1 = t_2 \in \Delta$ (this ensures constant domain and rigid designators); (2.2) R_c is as in the propositional case; (2.3) for all $t \in \mathcal{J}$, $V_c(t) = \{t^*: t = t^* \in \Delta\}$, and for all $Q \in \mathcal{R}^n$ and $w \in W_c$, $V_w(Q) = \{\langle V_c(t_1), \ldots, V_c(t_n)\rangle: Qt_1 \ldots t_n \in w\}$; (2.4) $D = \{V_c(t): t \in \mathcal{J}\}$.

The proof of strong model-completeness proceeds in the following three steps (this technique was suggested by Thomason (1970)):

STEP 1: LINDENBAUM–HENKIN-SATURATION-LEMMA Every **L**-consistent formula set Γ in language $\mathcal{L}Q1$ can be extended to an **L**-saturated formula set Δ in a language $\mathcal{L}Q1^*$ which differs from $\mathcal{L}Q1$ only in that it contains an additional denumerably infinite set C^* of *new* constants (i.e. $C^* \cap C(\mathcal{L}Q1) = \emptyset$, $C(\mathcal{L}Q1^*) = C \cup C^*$). Given an enumeration of all formulas $A_o, A_1 \ldots$ in $\mathcal{L}Q1^*$ and of all constants in C^*, one defines:

$$\Delta_o =_{df} \Gamma,$$

$$\Delta_{n+1} =_{df} \begin{cases} \Delta_n \cup \{A_n, \neg B[a/x]\} \text{ (where a is the first constant in } C^* - C(\Delta_n, A_n)), \\ \qquad \text{if } \Gamma_n \cup \{A_n\} \text{ is consistent and } A_n \text{ is of the form } \neg\forall xB \\ \Gamma_n \cup \{A_n\}, \text{ if } \Gamma_n \cup \{A_n\} \text{ consistent and } A_n \text{ is not of the form } \neg\forall xB \\ \Gamma_n \qquad \text{if } \Gamma_n \cup \{A_n\} \text{ is inconsistent} \end{cases}$$

$$\Delta =_{df} \cup \{\Delta_n : n \in \omega\}$$

For each n, there are infinitely many new constants remaining in $C^* - C(\Delta_n, A_n)$; thus the required new constant always exists. As in the non-modal case it is proved that Δ is **L**-saturated (Garson 1984: 271).

STEP 2 New in the quantificational case is the proof of the canonical model lemma. This lemma now assures the *existence* of a formula set which is not only maximally **L**-consistent, but also ω-complete *w.r.t. the same* language $\mathcal{L}Q1^*$ of Δ. For Q1-logics, this is proved by exploiting the Barcan formula.

CANONICAL Q1-MODEL LEMMA (1) If $\Gamma, \Sigma \subseteq \mathcal{L}Q1$, Γ is ω-complete, and ς is finite, then $\Gamma \cup \Sigma$ is ω-complete. (2) Every **L**-consistent and ω-complete formula set Γ can be extended to an **L**-saturated set Δ written in the *same* language (i.e. the language w.r.t which Γ was ω-complete). (3) If $\neg\Box B \in w \in W_c(\mathbf{L},\Delta)$, and w is **L**-consistent, then $\{A: \Box A \in w\} \cup \{\neg B\}$ is (3.1) **L**-consistent, (3.2) ω-complete, (thus) (3.3) has an **L**-saturated extension u written in the same language, such that (3.4) for all t_1, t_2: $t_1 = t_2 \in u$ iff $t_1 = t_2 \in \Delta$. (4) $\forall u \in W_c(\mathbf{L},\Delta)$: $\Box A \in u$ iff $\forall v \in W_c(uR_cv \Rightarrow A \in v)$.

PROOF *Exercise. Hints:* For 1 see Garson (1984: 274) (his lemma 1). For 2 see Garson (1984: 274f) (his lemma 2). The proof constructs Δ as above except that it shows that for each A_n of the form ¬∀xB, the required constant a exists in the old language, because Δ_n is already ω-complete by 1 of our lemma. The proof of our lemma 1 + 2 rests solely on classical quantifier principles. 3.1 is proved as in the propositional case. For 3.2, see Garson (1984: 275) (his lemma 3) – this proof depends on the Barcan formula. 3.3 follows from 3.1 + 2 by 2. For 3.4, see Garson (1984: 277f) – this proof uses the rigidity axiom (rI¬) and the theorem (rI). 4: this follows from 3 as in the propositional case.

STEP 3 It is now straightforward to prove the *Q1ML-Truth Lemma:* For every $A \in \mathcal{L}Q1^*$ and $w \in W_c$: $(M_c(\mathbf{L},\Delta), w) \models A$ iff $A \in w$. *Proof* by induction on formula complexity (Garson 1984: 275f; Hughes and Cresswell 1984: 84, 176). The atomic case holds by definition, the steps for propositional operators are as before; the only new item are the the following steps for the identity of formulas and quantifiers: Step 3.1: $t(M_c,w)\models t_1 = t_2$ iff $v_c(t_1) = v_c(t_2)$ iff $t_1 = t_2 \in \Delta$ (by definition of $v_c(t)$) iff $t_1 = t_2 \in w$ (by def. of W_c). Step 3.2: $(M_c,w)\models \forall xA$ iff $\forall d \in D_c$: $(M_c[x{:}d],w)\models A$, iff $\forall t \in \mathcal{J}(\mathcal{LQ}1^*)$: $(M_c[x{:}v_c(t)],w)\models A$ (by def. of D_c), iff $\forall t \in \mathcal{J}(\mathcal{LQ}1^*)$: $(M_c,w)\models A[t/x]$ (by coincidence lemma), iff $\forall t \in \mathcal{J}(\mathcal{LQ}1^*)$: $A[t/x] \in w$ (by induction hypothesis), iff $\forall xA \in w$ (by ω-completeness of w). Q.E.D.

Lindenbaum–Henkin and Truth Lemma establish as in the propositional case that:

Q1ML-MODEL-COMPLETENESS (1) Every normal Q1ML is strongly model-complete, and is strongly characterized by the class of its models. (2) **Q1K** is canonical.

As in the propositional case, to prove that a Q1-logic **L** stronger than **Q1K** is canonical requires to show that the frame of **L**'s canonical model is an **L**-frame. It is natural to *conjecture* that canonicity transfers from all *propositionally representable* Q1-logics to their Q1-counterpart. This conjecture was stated as an open problem in Hughes and Cresswell (1984: 183f) and was (wrongly) positively answered by Garson (1984: 276), But quite astonishingly, general canonicity transfer *fails*. An example of a canonical **L** $\in \Pi$ with a frame-incomplete Q1-counterpart is **S4.1**:

Q1S4.1-THEOREM (1) **S4.1** is canonical. (2) **Q1S4.1** is frame-incomplete. (3) **Q1S4.1** + $(\Diamond\Box\exists xA \supset \Diamond\exists x\Box A)$ is canonical.

PROOF For 1 see earlier. 2 is proved by showing that $\Diamond\Box\exists xA \supset \Diamond\exists x\Box A$ is valid on all **S4.1**-frames, but invalid in a certain nonstandard **Q1S4.1**-model; see Schurz (1997: 292f), the proof is due to Kit Fine. A proof of 3 is found in Schurz (1997: 293–5).

The reason why the proof of canonicity works for **S4.1** but *not* for **Q1S4.1** is that the first-order frame condition corresponding to **S4.1** contains an *existential* quantifier. This means in the propositional case that it has to be shown that a certain formula set has a maximally consistent extension, while in the predicate logical case it has to be shown that this formula set has a maximally consistent *and* ω-*complete* extension; but this is only possible if the additional axiom schemata $(\Diamond\Box\exists xA \rightarrow \Diamond\exists x\Box A)$ is available. However, the following restricted transfer theorem holds:

RESTRICTED Q1-CANONICITY-TRANSFER THEOREM (1) If a normal PML **L** = **KX** has the *subframe property* (which means that **L**'s frames are closed under subframes), then canonicity transfers from **KX** to **Q1KX**. (2) If **L**'s frames are definable by a *purely universal* first-order formula, then **L** has the subframe property.

The proof of 1 is based on the fact that the frame of M_c(**Q1KX**) is isomorphic with a subframe of M_c(**KX**) (see Schurz 1997: 295: for a similar result for intermediate logics see Shimura 1993: 36). The proof of 2 is straightforward. The theorem covers the axiom schemata D, T, 5, Alt_n, Ver, Triv, 0.3, because they correspond to universal first-order formulas; moreover it covers all subframe logics in the sense of Fine (1985: 624;

see Chagrov and Zakharyaschev 1997: 380ff) which include, among others, **KG** and **KGrz**. It is an open problem whether canonicity-transfer holds for larger classes of normal Q1MLs. In lack of stronger transfer results, canonicity has to be proved for each QML separately (see Gabbay (1976) and Bowen (1979) for various special canonicity results).

The transfer problem from monomodal to multimodal logics exists also in the quantificational case, but the propositional proof technique (§2.5) does not generalize to the quantified case. So far, only *canonicity transfer* from monomodal Q1-logics to their multimodal combination has been proved in Schurz (1997: 67).

Varying domains, rigid designators and free quantification: Q2-logics

The constant domain assumption implies that whatever exists in the actual world, exists necessarily, that is in all possible worlds. Moreover, for every $t \in \mathcal{J}$, 't exists' ($\exists x(x = t)$) is a theorem of nonmodal QL. Hence, $\Box\exists x(x = t)$ ('t necessarily exists') is a **Q1K**-theorem for every $t \in \mathcal{J}$. This idealization is inadequate when worlds are interpreted as possible states of the real world, because individuals do not have 'eternal' life. So there is a need to develop semantics with varying domains.

In models with world-relative domains, every world w has its own domain D_w of individuals – those objects which exist in world w. D_w is the range of the quantifier at w: $\forall xFx$ is true at w iff every $d \in D_w$ has property F. The Barcan formula $\forall x\Box A \supset \Box\forall xA$ is now invalid: it might well be that all individuals in D_w have the property F at all worlds v accessible from w, but some world v accessible from w has an individual in its domain D_v which is not in D_w and does not have property F at v (i.e. *not* $\forall v(Rwv \Rightarrow \forall d \in D_v((M[x{:}d],v) \models Fx))$. But also, the classical quantifier principles become problematic. Recall that the converse Barcan formula cBF $\Box\forall xA \supset \forall x\Box A$ is a theorem of every normal modal logic with classical quantifier principles. But in models with world-relative domains, cBF can only be valid if the condition of *nested domains* is satisfied: $uRv \Rightarrow D_u \subseteq D_v$. In order to keep classical quantifier principles, Hughes and Cresswell (1968: 171ff), Gabbay (1976: 44ff) and Bowen (1979: 8ff) adopt this condition.

The nested domain condition is rather restrictive. For symmetric R it even implies a *constant* domain for every generated model (recall syntactic Q1ML-theorem no. 5 in the previous subsection); so the difference to Q1-logics would vanish for all **QKB**-extensions with nested domains. But even *if* this condition is accepted, the classical quantifier principles are problematic, at least if designators are rigid. Assume $a \notin D_w$; for example, a = Pegasus and w = the real world. What truth value should be given to the sentence Fa, for example 'Pegasus has wings,' at world w? Since designators are rigid, the so-called requirement of *local* predicates, which says that only objects which exist at world w may be elements of predicate extensions at w, cannot avoid a conflict with classical quantifier principles. For, if $\forall xFx$ is true at worlds w, but $V(a) \notin D_w$ and $V(a) \notin V_w(F)$, then $\neg Fa$ and hence (by classical quantifier principles) $\exists x\neg Fx$ is true at w, contradicting the truth of $\forall xFx$ at w. A way out is to give sentences about nonexistents at w *no* truth-value at w. This leads to a semantics with *truth value gaps*, which has been developed by Hughes and Cresswell (1968: 170–3) and Gabbay (1976: 44ff).

If truth-value gaps should be avoided, we must allow that individuals may have properties at a world without being existent at world w. For example, we must allow that

'Pegasus has wings' is regarded as true at our world although Pegasus does not exist in our world. Classical quantifier principles can then no longer be valid, for $Fa \rightarrow \exists xFx$ comes out false at our world w. Hence, we must adopt the principles of *free logic*. I agree with Garson (1984: 261) that free QMLs are the most adequate choice for models with varying domains. In free logic, the classical UI-axiom is replaced by its *free* logic variant fUI: $\forall xA \supset (Et \supset A[t/x])$; in words: 'if all objects have property A, and t exists, then t has property A.' 'Et' is the *existence predicate*, defined as '$\exists x(x = t)$.' A like change is made for the rule UG. A first system of this kind has been suggested by Kripke (1963b); but Kripke only mentions this possibility (1936b: 70) and prefers to an axiomatization which avoids formulas with constants or free variables. Fine (1978) has given an elaboration of this kind of free modal QL for **S5**, and several further systems are discussed in Garson (1984: 257, 285). We call these logics **Q2**-logics and define their basic concepts as follows.

DEFINITIONS The Q2-language *ℒQ2* is syntactically like *ℒQ1*, but it is *interpreted* in different way. The *existence predicate* E in Q2-languages is *defined* by $Et =_{df} \exists x(x = t)$. An *Q2-model* (based on frame ⟨W,R⟩) is a quintuple M = ⟨W,R,U,Df,V⟩, where U Ø is the *total domain* of possible individuals and Df: W → Pow(U) is the *domain function* assigning to each world $w \in W$ its domain $D_w \subseteq U$. D_w is called the *inner* domain of w (the existing objects of w) and $U–D_w$ the *outer* domain of w (the nonexisting objects of w). (One could add the requirement $U = \cup_{w\in W}D_w$; but this would not bring new theorems; see Schurz (1997: 198).) The valuation function for terms and predicates and the truth clauses for atomic formulas, identity formulas, and propositionally compound formulas are as in Q1-logics. The only new clauses concern quantification: $(M, w) \models \forall xA$ iff for every $d \in D_w$, $(M[x:d],w) \models A$. This yields for the existence predicate: $(M,w) \models Et$ iff $v(t) \in D_w$.

The minimal normal Q2-logic, **Q2K**, is axiomatically defined like **K1** *except* that (1) the axiom schema BF is *dropped*, (2) the axiom UI is replaced by its *free* version fUI: $\forall xA \rightarrow (Ey \rightarrow A[t/x])$, and (3) the rule UG is replaced by its free version fUG: $Ex \rightarrow A/\forall xA$ (Garson 1984: 252; Fine 1978: 131f, suggests an equivalent axiomatization which keeps UR and adds '$\forall xEx$'). *Exercise:* Prove the duals fEI: $(Et \wedge A[t/x]) \supset \exists xA$, and fEG: $A \wedge Ex \supset B/\exists xA \supset B$, provided $x \notin \mathcal{V}_f(B)$. **L** is a normal Q2-logics, (**L** ∈ **Q2Π**) iff **L** extends **Q2K** and is closed under the rules of **Q2K** and under substitution for nonlogical predicates. As before, every **L** ∈ **Q2Π** is representable as **Q2KX**. The strategy of proving *model-completeness* which was used for Q1-logics *fails* for Q2-logics, because the Barcan formula is missing which allowed us to construct saturated sets in the same language. Fine (1978: 131–5) gives a proof of canonicity for **Q2S5** based on so-called *nice diagrams* (these are saturated sets of formula-world pairs). As far as I can see, this technique generalizes to all Q2-logics containing **Q2B**, but not to all Q2-logics. A general proof of model-completeness via a canonical model construction is possible by replacing the rule (fUG) by the following *stronger* rule. A *G-function* is a function G: *ℒQ2* → *ℒQ2* which assigns to each A ∈ *ℒQ2* a formula of the form $G(A) := B_0 \rightarrow \Box(B_1 \rightarrow \Box(B_2 \rightarrow \ldots \Box(B_n \rightarrow A) \ldots)$, for given $B_0, B_1, \ldots, B_n$ (n, 0) where B_0 may be missing. The stronger rule is:

GUG: $G(Ex \rightarrow Ax)/G(\forall xA)$, provided $x \notin \mathcal{V}_f(G(\forall xA)$

With minor simplifications I am following Garson (1984: 282ff; Garson also replaces fUI by GUI, but this replacement is redundant; see Schurz 1997: 199f). GUG preserves frame-validity and, thus, is correct w.r.t. the class of Q2-models. GUG also covers also rule (fUG*) (recall §3.1), and thus, it implies the axioms $\forall 1 + 2$. A Q2-logic where UG + $\forall 1 + 2$ are replaced by GUG is called a *QG2-logic*. Model-completeness of QG2-logics can be proved similar as for Q1-logics. The worlds of the canonical model $M_c(\mathbf{L},\Delta\}$ (in given $\mathcal{LQ}2^*$) are now all *G-saturated* formula sets; these are all maximally **L**-consistent formula sets Γ which are *G-complete*: $\Gamma \vdash_{\mathbf{L}} G(\forall xA)$ iff $\Gamma \vdash_{\mathbf{L}} G(Et \supset A[t/x])$ for every $t \in \mathcal{J}^*$. The proof proceeds through the same steps as before; due to the stronger G-rule it can be proved, without BF, that for every $w \in W_c$ with $\neg\Box B \in w$, $\{A: \Box A \subseteq w\} \cup \{\neg B\}$ can be extended to a G-saturated set in the same language. For terms and predicates, R_c and V_c are defined as before; $U_c = \{V_c(t): t \in \mathcal{J}^*\}$, Df_c: $W \rightarrow Pow(U_c)$ such that $D_c(w) = \{V_c(t): Et \in \Delta\}$. We thus obtain the *QG2ML-model-completeness-theorem:* Every QG2-logic **L** is correct and strongly complete w.r.t. the class of its Q2-models, and **Q2GK** is canonical.

Garson (1984: 284f) claims it as an open problem if and to which extent the rule GUG is indeed stronger than UG. Schurz (1997: 200) gives a partial answer, by proving the *GUG-Theorem:* In all Q2-logics which contain B, GUG is admissible. Hence, all normal extensions of **Q2B** are strongly model-complete. It is an open problem whether there exist model-incomplete Q2MLs that don't extend **Q2B**.

Concerning frame-completeness, the same restricted tranfer result as for Q1-logics can be proved for propositionally representable Q(G)2-logics. Schurz (1997: 201f) defines a translation function t: $\mathcal{LQ}2 \rightarrow \mathcal{LQ}1$ which translates Q2-formulas into semantically equivalent Q1-formulas, and Q2-models into corresponding Q1-models. An inverse translation is impossible: the $\mathcal{LQ}1$-quantifier figures like a *possibilistic* quantifier for translated $\mathcal{LQ}2$-logics; thus $\mathcal{LQ}1$ has greater expressive power than $\mathcal{LQ}2$. With the help of this translation function, various transfer theorems from Q1- to Q2-logics are established; in particular the following *frame-transfer:* for every $\mathbf{L} \in \mathbf{Q2}\Pi$: $\mathbf{F}(\mathbf{L}) = \mathbf{F}(t(\mathbf{L}))$, where $t(\mathbf{L})$ is the Q1-translation of Q2-logic **L**. If **X** is propositional, then $t(\mathbf{X}) = \mathbf{X}$; hence propositionally representable Q2-logics have the same frame-classes as their Q1-counterparts. Whether transfer of frame-completeness from Q1MLs to Q2MLs is possible remains an open problem (Schurz 1997: 204).

Nonrigid designators, counterpart theory, and worldline semantics: Q3-logics

Rigid designators presuppose that the fixation of their reference does not depend on any contingent property of the individual to which they refer. This may be true for mathematical objects such as '7' or '9' (Kripke 1972 uses them often as illustrations), but is it possibly true for empirical objects? According to Putnam's famous account of meaning (1975), the fixation of rigid reference is based (1) on an indexical relation of direct acquaintance with the individual in the present (*hic et nunc*) state (the act of 'babtizing'), and (2) on a unique relation of causal successorship or predecessorship. Accordingly, there are two problems with that account. Concerning (2), nothing guarantees that the relation of predecessor- or successorship in past and future states is uniquely determined. Take Frege's old example of the morning and the evening star, both of which are identical with the planet Venus: assume that in some future time, Venus splits into two planets, one appearing only at the morning and the other in the evening, then, to what objects will the names 'morning star' and 'evening star' refer in that distant future state? (A more realistic example is the process of cell division.) And concerning (1), the relation of 'acquaintance' in the act of 'baptizing' is never absolutely 'direct' but always mediated through contingent properties.

Hintikka (1961) and Kanger (1957b) have already made suggestions for QMLs with non-rigid designators, in short: *nonrigid* QMLs (also see Hughes and Cresswell 1968: 195). Syntactically, the axiom rI¬ has to be dropped for nonrigid QMLs, and the rigid principle of substitution of identicals rISub must be restricted to *nonmodal* formulas as follows:

ISub: $t_1 = t_2 \supset (A[t_1/x] \supset A[t_2/x])$, provided A does not contain '$\Box$'.

As a result, the identity theorems of nonrigid QMLs are no longer closed under the unrestricted rule of substitution for predicates. They are still closed under substitution of arbitrary nonmodal formulas for predicates (cf. Schurz 1997: 221).

Semantically, nonrigid designators require a world-relativization of the valuation functions for terms; v: $\mathcal{J} \times W \to U$ where $v(t,w) =_{df} v_w(t)$ is the extension of term t at world w, and the partial function $v(t): W \to U$ such that $v(t)(w) = v_w(t)$ is the *intension* of term t. The debate between Kripke and Lewis, whether individuals in different worlds are strictly identical (Kripke) or merely counterparts of each other (Lewis), is logically less decisive than one might think. Rigid designator axioms are also adequately characterized by the *unique counterpart view*, according to which every individual possesses a unique counterpart in every possible world (see also Forbes 1985: 60ff). We just have to assume that the valuation function v assigns to each term t and world w a pair $\langle d,w\rangle$, which stands for the world-relativized individual *d-in-w*, such that the domain component 'd' of this pair is the same in all worlds. Then, each world w has its own domain $D \times \{w\}$ and each world-relativized individual $\langle d,w\rangle$ has a unique counterpart $\langle d,u\rangle$ in each world $u \in W$. The resulting logic would be Q1ML (but the same modification can be made for Q2MLs). Hence, the important point of a semantics for nonrigid designators, which do not obey rigid identity axioms, is the assumption of a counterpart relation which is *not unique*.

The real problem of nonrigid designators is the semantical interpretation of *quantified de re formulas*. Take, for example, $(M,w) \models \exists x\Box Fx$. This means formally that there exists $d \in D_w$ such that for all w-accessible worlds v: $(M[x:d],v) \models Fx$. But how do we define the x-variation V[x:d] of V^M if designators are non-rigid? The most simple possibility would be to assume that V[x:d] assigns d to x in *all* worlds. In the effect, this means that variables are interpreted as *rigid* designators; only constants are nonrigid. This option is chosen by Thomason (1970). However, the free quantification axiom (fUI) becomes invalid in these systems: from the (fEG)-instance $\Box(t = t) \wedge Et \supset \exists x\Box(x = t)$ the formula $Et \supset \exists x\Box(x = t)$ is provable, though it is invalid, because it requires t to have the same extension at all accessible worlds, which need not be the case (Garson 1984: 262). Hintikka (1970) suggests to replace (fUI) by a complicated instantiation rule, which in case of Thomason's system **Q3-S4** reduces to $\forall xA \supset (\Box Et \supset A[x/t])$ (Garson 1984: 263); generalized completeness proofs for these kinds of systems have not been found. A possibility of handling systems with rigid variables and nonrigid constants, elaborated by Bowen (1979), is to assume that terms are *local*, that is that their extensions at worlds always exist in that world; this locality option becomes available when terms are nonrigid. Bowen also accepts the nested domain condition, with the result that classical quantification principles are valid in his systems, and generalized proofs of model-completeness are possible.

All systems with rigid variables contain the theorem $\forall x\forall y(x = y \supset \Box x \equiv y)$, which a strict defender of nonrigidity wants to avoid. Another possibility of defining v[x:d] would be to allow V[x:d] to be any function from W into D, satisfying only the restriction that $V_w(x) = d$. In counterpart terminology, this means that anything may count as a counterpart of d in other worlds. This semantics corresponds to Garson's *conceptual* interpretation (1984: 266). Apart from the resulting incompleteness (Garson 1984: 266), this semantics validates the counterintuitive formula $\Box\exists xFx \supset \exists x\Box Fx$, which is a clear reason to reject it (Hughes and Cresswell 1968: 197f).

What one needs is a way to *restrict* the 'allowed' functions over which v[x:d] may range, and the natural way to do this is by way of a *counterpart relation* which specifies the counterparts of $d \in D_w$ in all w-accessible worlds. This account has been developed by Lewis (1968), though not within the framework of modal logic but within that of ordinary first-order logic, and by assuming universal frames. Specifically, Lewis introduces the predicates Ww for 'w is a world,' 'Ixw' for 'object x exists in world w,' and Cxy for 'y is a counterpart of x.' The counterpart relation need neither be symmetric nor transitive. Let us present Lewis' theory in the framework of modal logic, interpreting Cxy as 'y is a counterpart of x in a world w accessible from x's world.' Then Lewis' proposed semantical interpretation of *de re* sentences can be reformulated in this way (1968: 118):

$(M[x_{1-n}:d_{1-n}],u) \models \Box A$	iff for all w-accessible worlds v and *all* $d'_1, \ldots, d'_n$ such that each d'_i is a counterpart of d_i in v, $(M[x_{1-n}:d'_{1-n}],v) \models A$.
$(M[x_{1-n}:d_{1-n}],u) \models \Diamond A$	iff for some w-accessible worlds v and *some* $d'_1, \ldots, d'_n$ such that each d'_i is a counterpart of d_i in v, $(M[x_{1-n}:d'_{1-n}],v) \models A$.
For terms: $(M,u) \models$	$\mathbf{o}A[t_1/x_1, \ldots, t_n/x_n]$ iff $(M[x_{1-n}:v(t_{1-n})],u) \models \mathbf{o}A$, with $\mathbf{o} \in \{\Box,\Diamond\}$; i.e., iff for all/some w-accessible worlds v and all/some $d_1, \ldots, d_n$ such that each d_i is a counterpart of t_i's extension at w in v, $(M[x_{1-n}:d'_{1-n}],v) \models A$.

The problem is that Lewis' counterpart theory, if taken as a semantics for modal logic, is logically not well-behaved. It is not closed under substitution, even not substitution of atomic formulas for propositional variables. For example, $\Box p \supset \Box\Box p$ will be valid on a transitive frame, yet $\Box Ft \supset \Box\Box Ft$ might be invalid in a Lewis model imposed on that frame, because the counterpart relation need not be transitive. More drastically, Wollaston (1994) shows that Lewis' semantics invalidates the modal principles K and M, and even the nonmodal principle UI. Ghilardi (1991) has developed a semantics for nonrigid QML which adopts the nested domain condition and models counterpart relations as functions $c: D_u \to D_v$ for uRv. His systems are logically well-behaved, but he obtains drastic incompleteness results; for example the QML $\mathbf{Alt_n}$ is incomplete in his semantics, though canonical in our Q1-, Q2- and Q3-semantics (cf. corollary 7.5 of

Ghilardi). More recently, Skvortsov and Shehtman (1993) have introduced a new kind of frame semantics, so-called *metaframe semantics*, which is a generalization of Ghilardi's functor semantics. They are able to show that completeness w.r.t. metaframes generally transfers from a PML to its quantificational counterpart. Technically, this is a great success. However, metaframe semantics is *not* based on domains of individuals, but on domains of 'abstract' n-tuples which are not reducible to the nth Cartesian product of an ordinary domain. So far, no one has given a philosophically transparent interpretation of metaframe semantics.

The philosophically more transparent alternative is the *substantial* interpretation of quantifiers, where quantifiers do not range over objects (term extensions), but over functions from worlds into objects (term intensions). This suggestion has been introduced by Hughes and Cresswell (1968: 198ff) and is extensively elaborated in Garson (1984: 267ff); specifically in his system **QS**. One assumes here, for each world, a set of term intensions, that is functions from W to D, which are the 'substances' which exist at that world; quantifiers range over these term intensions. Schurz (1997) shows that with some modifications, Garson's semantics can be reinterpreted from the *objectual* view as a certain kind of counterpart semantics, so-called *world-line semantics*. We call the logics based on it Q3-logics, and explain it as follows.

A 'term-intension,' that is a function $l{:}W \rightarrow U$ is called a *worldline* (in analogy to wordlines in Minkowski's space–time diagrams). A worldline l *lands* object d at world w if $l(w) = d$. The important component of Q3-models is a set L of worldlines ('substances') which specifies the possible term intensions. L determines a four-placed counterpart relation 'object d_1 in u has d_2 as a counterpart in v' defined as follows: there exists a worldline in L which lands d_1 at u and d_2 at v. For each $w \in W$, U_w is the set of objects landed by some wordline at w, that is, the set of all w-counterparts of *possible* objects. Predicate extensions at w are taken from U_w. To obtain a *free* logic version of this semantics we also need a domain function Df which assigns to each world w a subset $D_w \subseteq U_w$ of objects *existing* in w. The world-specific sets of worldlines L_w over which quantifiers range are given as the set of worldlines which land some object in D_w.

Q3ML-DEFINITIONS The Q3-language $\mathcal{LQ}3$ is syntactically like an $\mathcal{LQ}1$-language; the existence predicate E is defined as in $\mathcal{LQ}2$. A Q3-model based on a frame $\langle W,R\rangle$ is a 6-tupe $\langle W,R,L,U,Df,V\rangle$, with $\emptyset \neq L \subseteq \{l{:}W \rightarrow U\}$ a nonempty set of possible worldlines, where $U \neq \emptyset$ is a nonempty set of possible objects; Df: $W \rightarrow U$ such that $Df(w) =_{df} D_w \subseteq U_w$ is the domain function, where $U_w =_{df} \{d \in U{:}\ \exists l \in L(l(w) = d)\}$ is the set of term-extensions at w. We define $L_w =_{df} \{l \in L{:}\ \exists d \in D_w(l(w) = d)\}$. Concerning V: for each $t \in \mathcal{J}$, $V(t) \in L$; and for each n-ary $Q \in \mathcal{R}^n$, $V_w(Q) \subseteq {U_w}^n$. M[x:l] denotes a model which is like M except that it assigns the wordline l to x. The truth clauses are as follows: (i) $(M,w) \models Qt_1 \ldots t_n$ iff $\langle V_w(t_1), \ldots, V_w(t_n)\rangle \in V_w(Q)$; $(M,w) \models t_1 = t_2$ iff $V_w(t_1) = V_w(t_2)$; for propositional operators as before; and for the quantifier: $(M,w) \models \forall xA$ iff for all $l \in L_w$, $(M[x{:}l],w) \models A$; this yields $(M,w) \models Et$ iff $V_w(t) \in D_w$ for the existence predicate.

Worldline semantics is fully compatible with the objectual view. Identity and existence statements depend only on the extensions of terms. The truth clauses for quantifiers may be rephrased in Lewis' counterpart style where quantifiers range over objects as follows: $(M,w) \models \forall xA$ [$\exists xA$] iff for all [some, resp.] $d \in D_w$ and $l \in L$ such that

$V_w(l) = d$: $(M[x{:}l],w) \models A$. For each particular formula, the quantification over worldlines is eliminable, for example: $(M,u) \models \forall x\Box Fx$ $[\exists x\Diamond Fx]$ iff for all [some, resp.] $d \in D_w$, w-accessible world(s) v, and v-counterpart(s) d' of d: d' is in $V_v(F)$.

The essential difference to Lewis' counterpart semantics is threefold. First, the counterpart relation defined by worldlines is symmetric and obeys further structural properties which are not satisfied by Lewis' counterpart relation. Second, quantification over counterparts is in worldline semantics governed by the quantifier, but in Lewis' semantics governed by the modal operator. The *de re* formulas $\forall x\Box Fx$ and $\exists x\Diamond Fx$ are evaluated in the same way, but the *de re* formulas $\forall x\Diamond Fx$ and $\exists x\Box Fx$ are evaluated differently: in Lewis semantics, $(M,w) \models \forall x\Diamond Fx$ iff for every $d \in D_w$ there exists *some* w-accessible world u such that *some* counterpart d' of d in u is in $V_u(F)$, while in worldline semantics, $(M,w) \models \forall x\Diamond Fx$ iff for every $d \in D_w$ there exists *some* w-accessible world u such that *every* counterpart d' of d in u is in $V_u(F)$. Likewise for the formula $\forall x\Box Fx$. Third, Lewis' semantics does not assign wordlines (term intensions) to terms t, but quantifies over the counterparts of term extensions $V_w(t)$ in de re scopes, while wordline semantics determines the counterparts of $V_w(t)$ by their wordlines l(t). For example, assume Dudu is the name of an amoeba a at world w which in all accessible worlds u splits up into two, namely b and c, where b keeps alive and c is dying in u, and we decide that Dudu should name b (but not c) at all w-accessible u. Then the sentence 'necessarily Dudu is alive' is true in worldline semantics, but false in Lewis style counterpart semantics. Note finally that the language of worldline semantics has a greater expressive power than that of Lewis' counterpart semantics. Lewis' modal operators ($\Box_L$, $\Diamond_L$) are *definable* within worldline semantics as follows (Schurz 1997: 222):

$$\Box_L A[t_{1-n}/x_{1-n}] =_{df} \forall y_{1-n}(\wedge\{t_i = y_i\text{: } 1 \le i \le n\} \supset \Box A[y_{1-n}/x_{1-n}]), \text{ and}$$
$$\Diamond A[t_{1-n}/x_{1-n}] := \exists y_{1-n}(\wedge\{t_i = y_i\text{: } 1 \le i \le n\} \wedge \Diamond A[y_{1-n}/x_{1-n}]), \text{ where } x_1, \ldots, x_n = \mathcal{T}(A).$$

The essential difference of worldline semantics as compared to Garson's substantial semantics is that Garson does not define the world-relative sets of worldlines ('substances') L_w by the extension of an ordinary existence predicate, as we did, but he introduces them directly, without such a predicate, and the truth clause of his existence predicate is: $(M,w) \models Et$ iff $v(t) \in L_w$ (Garson 1984: 279). This turns his existence predicate into an 'intensional' one which contains as its world-specific extension a set of term intensions. As a result, substitution of E in the identity axiom (ISub) is not allowed in Garson's system (1984: 268); though it is allowed in our system. Besides this greater simplicity, it seems to be philosophically more intuitive *not* to assume world-specifics set L_w as a *primitive* notion, for the existence of worldlines ('substances') is not a contingent matter; only the existence of objects is contingent.

The logic **Q3K** is defined like **Q2K** except that the rigid identity axiom rI¬ is dropped and rISub is replaced by ISub above. ISub is only closed under restricted substitution for predicates, while the other axiom schemata are closed under general substitution. On this reason, normal Q3-logics cannot be defined as before. We rather have to define a normal Q3-logic as a subset $\mathbf{L} \subseteq \mathcal{LQ}3$ which is representable as **Q3KX**, that is it contains all axioms (not merely the schemata) of **Q3K**, is closed under the rules of **Q3KX** (TautR, fUG, N) and contains all (unrestricted substitution) instances of the additional set of axiom schemata **X**, except for additional identity axioms in **X** to which only nonmodal substitution applies (Schurz 1997: 221). The canonical model $M_c(\mathbf{L})$ of a Q3ML **L** need no longer be relativized to an initial saturated set Δ which determines rigid identities. It is defined as $\langle W_c, R_c, L_c, U_c, Df_c, V_c \rangle$, where W_c is now the set of all G-saturated formula sets, R_c is an s usual, and $V_c(t)$: $W_c \rightarrow U_c$ is such that $V_{c,w}(t) = \{t' \in \mathcal{J}^*: t = t' \in w\}$, $U_c = \{V_{c,w}(t): w \in W, t \in \mathcal{J}^*\}$, $L_c = \{V_c(t): t \in \mathcal{J}^*\}$, Df_c:$W_c \rightarrow Pow(U_c)$ such that $Df(w) = \{V_c(t): t \in \mathcal{J}^*, Et \in w\}$. Model-completeness is proved with help of the stronger G-rule GUG in the same way as for Q2-logics (Garson 1984: 282ff). We thus arrive at the *QG3ML-model-completeness-theorem*: all normal QG3MLs are adequately characterized by the class of their models, and **Q3K** is canonical. Q(G)3-logics behave similar as Q2-logics: we can show that GUG is admissible in all normal extensions of **Q3B**, that restricted canonicity-transfer holds, and that the frames of Q3-logics are the same as their Q1-counterparts (Schurz 1997: ch. 10.8–10). A different technique proves model-completeness for Q3-logics by introducing for each canonical world a new set of constants. This proof avoids the stronger G-rules, but it is not completely general: certain properties of the canonical frame cannot be proved in the standard way because canonical worlds don't share the same language (Garson 1984: 276–81).

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