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First-Order Alethic Modal Logic

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1 Introduction

Propositional modal logic, with its possible world semantics, is now a standard part of a philosophical education, while first-order modal logic is less familiar. But there are several well-known problematic concepts that can be made more intelligible using a first-order modal semantics; among these are existence, designation, identity, synonymy, intension, and extension. I will address these and other issues. I will assume a general familiarity with propositional possible world semantics, and begin at the quantificational level; (Hughes and Cresswell 1996) is a standard reference. I will not attempt to move from semantics to proof procedures, length precludes that, but (Fitting and Mendelsohn 1998) contains tableau systems that are appropriate for what is presented here.

I should begin by saying something about the status of possible worlds. It is sometimes asked what they are, or even where they are. These are the wrong questions to be asking. Consider classical logic, for a moment. To say a formula Φ is valid is to say it is true in all models. One does not inquire where these models come from – we are talking formal mathematics, and they exist in the same sense that any mathematical structure exists. (I grant that questions of mathematical existence can be tricky too, but they are not what concern us now.) In addition, we occasionally apply classical logic to the actual world – we extract a formal model from 'reality.' When we do so we must stipulate the domain of quantification. This amounts to specifying what the 'things' of the real world are. Do they include numbers? Do they include concepts like beauty? Applying classical logic to the real world is not as straightforward as we often make it seem, but nonetheless, we do it.

Modal models involve possible worlds. Generalizing from classical logic, a formula is taken to be valid if it is true no matter what the domain and no matter what the interpretation of symbols, *and no matter at what possible world of a model we evaluate the formula*. This is a formal definition, just as in the classical case. Possible-world models are mathematical structures too.

We still must deal with the desire to apply modal notions in the actual world. The problem is much like that of applying classical logic to the actual world but now, in addition to stipulating domains and interpretations, we must also stipulate possible worlds. They are not ‘out there’ to be found with a telescope. Intuitively, they represent how things might have been, and to a considerable extent, this is up to us. Is a situation in which Julius Caesar was a bottle of salad dressing really a way things could have been, or not? It does not seem to me that such a question has an answer independent of the asker, just as whether beauty is in the range of a quantifier or not probably depends on who is using the quantifier, and for what purpose. In short, as a piece of mathematics, possible world semantics is on the same footing as all mathematics. As a way of understanding discourse about the real world, the semantics goes a long way towards clarifying things, but there is considerable ambiguity or, if you prefer, flexibility.

In what follows I will sometimes be describing formal models, mathematical structures. But sometimes I will be using possible-world semantics informally, with some intuitive notion of possible worlds which I assume is sufficiently understood by both me and the reader to make the discussion mutual. In such situations, the real world will generally be assumed to be among the possible worlds, and the quantification domain will be assumed to include at least all real things. But keep in mind the discussion above as to what is a real thing. By keeping the discussion imprecise I am, in effect, allowing for a variety of different ways of understanding everyday modal discourse in terms of possible-world semantics.

2 Intensions

Let us say an adult is someone 21 or older. The property of being an adult has a certain *extension*: the set of people who are, in fact, 21 or older. At other times, or under imagined circumstances, the same property will have a different extension. The *intension* of the property is, in some indefinite sense, its meaning, and so determines its extension under various circumstances. Trying to formalize meaning is a formidable task, and reasonable people can differ about how this should be done. The common denominator among all such attempts is: the intension of a property should determine its extension, in every circumstance. If we ignore the issue of how, intensions simply become maps from situations to extensions.

In addition to properties, we also need to treat individuals and individual concepts. The number 9, and Bertrand Russell, are individuals, or individual objects. The number of the planets, or the junior author of the *Principia*, are individual concepts. As things are, they designate 9 and Bertrand Russell respectively, but under other circumstances they might not have done so. Once again, some notion of meaning is involved. And once again, however that notion of meaning is understood, an individual concept will associate an individual object with each circumstance. Formalized it will simply be a map from situations to objects.

This leads to the beginnings of a formal treatment – (Fitting and Mendelsohn 1998) contains a fuller version of what follows. I’ll assume we have a first-order modal language with *relation symbols* of various arities. The equality symbol, =, is among them. (Since it’s what we’re used to, I’ll write = in the conventional infix position.) There will also be *constant symbols* – typically c, d, . . . And there will be *variables* – typically x, y, . . . Relation symbols will be used to represent properties in *intension*, and constant symbols will be used to represent *individual concepts*. Intensions determine extensions, which are sets of objects, and likewise individual concepts determine objects, so we need machinery for dealing with objects as well. I'll assume variables have individual objects as values. For an atomic formula, say $P(c)$, it will be taken to be true at a possible world if the individual object designated by c at that world is in the extensional property designated by P at that world.

There is yet one more piece of machinery that must be introduced, and it will be less familiar. In *classical* logic, if $\Phi(x)$ is a formula, we can think of it as determining the extensional property of being something that makes Φ true. But now we are trying to think intensionally. Using $\Box$ for the necessity symbol and P as a one-place relation symbol, how should we understand the formula $\Box P(x)$, that is, what *intensional* property does it determine? In particular, for a constant symbol c, how should we read the formula $\Box P(c)$? Should it be taken to say c has the P property necessarily, or that c has the necessary-P property? These are *not* synonymous. Suppose, for instance, that c is the richest-person-in-the-world individual concept, and P is the intensional property being-wealthy (both notions change with changing circumstances). It seems likely that $P(c)$ is true under all circumstances – the richest person in the world, whoever that is, is wealthy, however we measure wealth. Then $\Box P(c)$ should be taken to be valid, since $P(c)$ is always true. On the other hand, while c designates the richest person in the world currently, that person might be poor under other circumstances, so we cannot say, of c, that the person has the necessarily-wealthy property. But then $\Box P(c)$ should not be taken to be valid. What is needed is some way of distinguishing between these two interpretations of the single formula $\Box P(c)$.

I'll make use of a device called *predicate abstraction*. If Φ is a formula and x is a variable, $\langle \lambda x.\Phi \rangle$ is a predicate abstract. If t is a term – either a constant symbol or a variable – and $\langle \lambda x.\Phi \rangle$ is a predicate abstraction, $\langle \lambda x.\Phi \rangle(t)$ will be counted as a formula. Then $\Box\langle \lambda x.P(x)\rangle(c)$ and $\langle \lambda x.\Box P(x)\rangle(c)$ are both formulas, and obviously different. The semantics introduced below will give them different readings, corresponding to the two readings of $\Box P(c)$ above.

Now the class of *formulas* can be specified. It is built up in more-or-less the usual way, using propositional connectives $\wedge$, $\vee$, $\supset$, $\equiv$, and $\neg$, modal operators $\Box$ and $\Diamond$ quantifiers $\forall$ and $\exists$, and predicate abstraction. I skip details, as they are quite straightforward to supply. For simplicity, I'll abbreviate formulas like $\langle \lambda x.\langle \lambda y.\langle \lambda z.\Phi \rangle(e)\rangle(d)\rangle(c)$ by $\langle \lambda x, y, z.\Phi \rangle(c, d, e)$.

3 Models

A *frame* is a structure $\langle \mathcal{G}, \mathcal{R} \rangle$, where $\mathcal{G}$ is a (nonempty) set of *possible worlds* and $\mathcal{R}$ is a binary relation on $\mathcal{G}$ of *accessibility*. Intuitively, one thinks of the members of $\mathcal{G}$ as representing the way things are, and the various ways they could be – possible situations, say. The accessibility relation tells us which situations are relevant to which. It is, by now, common knowledge that placing natural restrictions on $\mathcal{R}$ produces well-known modal logics. In many ways, S_5 is the simplest of the modal logics, and the most natural if $\Box$ is to represent metaphysical necessity. For S_5 $\mathcal{R}$ is simply the universal relation, the one that always holds. In what follows, I'll assume this is my choice for $\mathcal{R}$. Certain things are simpler and more natural with such a choice, though much of what is said applies more generally – see (Fitting and Mendelsohn 1998).

Certainly if things were different, different things might exist. An *extended frame* is a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame and $\mathcal{D}$ is a *domain function* mapping $\mathcal{G}$ to nonempty sets. If $\Gamma \in \mathcal{G}$, think of $\mathcal{D}(\Gamma)$ as the set of objects existing in Γ. Also, by the *domain of the frame* I mean the union of the domains of the various possible worlds. If we understand possible existence to mean actual existence under other circumstances, the domain of a frame consists of those things having actual or possible existence in our formal setting.

A *model* is a structure $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{I}$ is an *interpretation* in the extended frame $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$. The *domain of the model* is the domain of the underlying frame. The interpretation must meet three requirements. First, it should associate with every constant symbol c a mapping, $\mathcal{I}(c)$, assigning to each possible world some member of the domain of the model. Second, it should associate with every n-place relation symbol R a mapping, $\mathcal{I}(R)$, assigning to each possible world some n-place relation on the domain of the model. Third, it should associate with the equality symbol, =, the constant mapping assigning to each possible world the equality relation on the domain of the model.

The notion of interpretation captures the informal idea expressed earlier. Associated with each relation symbol is a relation in intension – a map from possible worlds to relations in extension. Likewise, associated with each constant symbol is an individual concept. Say we have a constant symbol c and a possible world Γ. $\mathcal{I}(c)$ is a function on possible worlds, so $\mathcal{I}(c)(\Gamma)$ is some member of the domain of the model. There is no requirement that it be something that exists at Γ; that is, it need not be in $\mathcal{D}(\Gamma)$. It makes perfectly good sense to talk about Pegasus, who exists in a mythological world even though he does not exist in ours. Similarly a relation symbol, at a world, is some relation in extension, but there is no requirement that things in that relation in extension actually exist at that world. If there were such a requirement, we would be unable to say that Pegasus has the property of being mythological.

4 About Quantification

If I claim that everything has a certain property, what am I claiming? I could mean everything that *actually exists* has the property (actualist quantification). I could mean everything that *does or could exist* has the property (possibilist quantification). In our formal semantics, actualist quantifiers, at a world Γ, range over the domain of that world, $\mathcal{D}(\Gamma)$. Possibilist quantifiers range over the domain of the model. Both are natural, but for different purposes.

Here, possibilist quantification will be taken as basic, because there is an easy way to define actualist quantification from it. Introduce a special one-place relation symbol, E, and interpret it at each world as the set of things that actually exist there – an existence predicate, in other words. Formally, in a model $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$, we will require that $\mathcal{I}(\mathsf{E})$ be the function that maps each possible world Γ to $\mathcal{D}(\Gamma)$. Further, introduce relativized quantifiers: $(\exists^{\mathsf{E}}x)\Phi$ abbreviates $(\forall x)[\mathsf{E}(x) \supset \Phi]$ and $(\exists^{\mathsf{E}}x)\Phi$ abbreviates $(\exists x)[\mathsf{E}(x) \wedge \Phi]$. Intuitively speaking (since the full formal semantics has not been fully specified yet), if $(\forall x)$ and $(\exists x)$ are read in a possibilist way, quantifying over the domain of the model, then $(\forall^{\mathsf{E}}x)$ and $(\exists^{\mathsf{E}}x)$ correspond to actualist quantification, with things restricted to world domains.

5 Truth in Models

Now comes the key definition: truth of formulas at possible worlds of a model. Simultaneously, the meaning of predicate abstracts must also be defined. Formulas can contain free variables, and so we need machinery for giving them values. A *valuation* is a mapping from free variables to the domain of a model. Note that valuations do not depend on possible worlds – free variables are supposed to represent objects, not intensions. If v is a valuation and $\mathcal{I}$ is an interpretation, between them they supply meanings for all terms. I'll use the following notation. For a possible world Γ,

(1) If x is a variable, $(v * \mathcal{I})(x, \Gamma) = v(x)$.
(2) If c is a constant symbol, $(v * \mathcal{I})(c, \Gamma) = \mathcal{I}(c)(\Gamma)$

Thus for any term t, $(v * \mathcal{I})(t, \Gamma)$ is the object associated with t at possible world Γ. I'll also use the following notation. If v is a valuation, x is a variable, and d is an object in the domain of the model, $v[x/d]$ is the valuation that is like v except that it maps x to d. And I'll say a formula is an *atom* if it is of the form $R(t_1, \ldots, t_n)$ where R is an n-place relation symbol and $t_1, \ldots, t_n$ are terms, or if it is of the form $\langle\lambda x.\Phi\rangle(t)$ where $\langle\lambda x.\Phi\rangle$ is a predicate abstract and t is a term.

Now, the fundamental notion to be defined is symbolized $\mathcal{M}, \Gamma \Vdash_v \Phi$ and is read: formula Φ is true at possible world Γ of model $\mathcal{M}$ with respect to valuation v. Simultaneously meanings are assigned to predicate abstracts. Here is the definition.

Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model.

(1) For atoms, $\mathcal{M}, \Gamma \Vdash_v R(t_1, \ldots, t_n)$ if $\langle (v * \mathcal{I})(t_1), \ldots, (v * \mathcal{I})(t_n) \rangle \in \mathcal{I}(R)(\Gamma)$.
(2) $\mathcal{M}, \Gamma \Vdash_v (X \wedge Y)$ if $\mathcal{M}, \Gamma \Vdash_v X$ and $\mathcal{M}, \Gamma \Vdash_v Y$, and similarly for the other propositional connectives.
(3) $\mathcal{M}, \Gamma \Vdash_v \Box X$ if $\mathcal{M}, \Delta \Vdash_v X$ for every $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, and similarly for $\Diamond X$.
(4) $\mathcal{M}, \Gamma \Vdash_v (\forall x)X$ if $\mathcal{M}, \Gamma \Vdash_{v[x/d]} X$, for every x in the domain of the model $\mathcal{M}$, and similarly for $(\exists x)$.
(5) The interpretation $\mathcal{I}$ is extended to predicate abstracts as follows. $\mathcal{I}(\langle\lambda x.\Phi\rangle)$ is the map that assigns to possible world Γ the set $\{d \mid \mathcal{M}, \Gamma \Vdash_{v[x/y]} \Phi\}$.

The definition is technical, but the content is intuitive. Item 1 says an atom is true at a world if the individual objects associated with the subject terms, at that world, are in the extension of the predicate, at that world. Item 5 says the intension of a predicate abstract, in a model, is determined in the obvious way by the behavior of the formula being abstracted. The other items are essentially standard.

6 Equality

Now that the technical definitions have been given, it is time to see how things behave. I'll begin with equality, whose interaction with necessity has always been considered a bit tricky. Recall, $\mathcal{R}$ is taken to hold between any two worlds in our discussion, so the underlying logic is S_5.

Suppose c and d are constant symbols, so that $c = d$ is a formula. To say it is true at a possible world of a model is to say the interpretations of c and d, at that world, are in the interpretation of = at that world. Since the interpretation of = is the equality relation at every possible world, this amounts to saying that c and d designate the same object at the world. Formally we have the following: $\mathcal{M}, \Gamma \Vdash_v c = d \Leftrightarrow \mathcal{I}(c)(\Gamma) = \mathcal{I}(d)(\Gamma)$.

What about necessary equality? If we say of two individual concepts that they are necessarily equal, are we saying their equality is necessary, or are we saying they have a 'necessarily equal' property. That is, are we asserting $\Box\langle\lambda x, y.x = y\rangle(c, d)$ or are we asserting $\langle\lambda x, y.\Box(x = y)\rangle(c, d)$? The two are not synonymous.

Consider first $\Box\langle\lambda x, y.x = y\rangle(c, d)$, or equivalently $\Box(c = d)$. To say this is true at possible world Γ of a model is to say $c = d$ is true at every world. By the analysis above, this amounts to saying c and d designate the same object at every world. This is a strong requirement, and it really amounts to saying c and d are *synonymous*. It is easy to produce formal models in which $(c = d) \supset \Box(c = d)$ is not valid, that is, in which $(c = d) \supset \Box\langle\lambda x, y.x = y\rangle(c, d)$ is not valid.

Now consider the other version, $\langle\lambda x, y.\Box(x = y)\rangle(c, d)$. Suppose that c and d happen to designate the same object at possible world Γ of a model. Certainly, at every world, that object is identical to itself. But this is just what it takes for $\langle\lambda x, y.\Box(x = y)\rangle(c, d)$ to be true at Γ. Thus $(c = d) \supset \langle\lambda x, y.\Box(x = y)\rangle(c, d)$ is simply a valid formula.

The difference between the two versions is striking, at least until one realizes that different things are really being said. We might read $\Box\langle\lambda x, y.x = y\rangle(c, d)$ as asserting it is necessary *that* c and d be equal. This is an assertion about their intensions and, as noted above, really asserts synonymy. Likewise we might read $\langle\lambda x, y.\Box(x = y)\rangle(c, d)$ as asserting the necessary equality *of* c and d, that is, of the objects designated by c and d. Well, if *objects* are equal under any circumstances, they cannot be otherwise and so we have, *of* c and d, that their equality implies their necessary equality.

Suppose we apply these observations to a few well-known problematic cases, discussed in Quine (1953a). Say we have a model in which the possible worlds include the actual one and various alternatives to it – representing how things could have been. Let 'n' be a constant symbol intended to be interpreted as *the number of the planets*, which can vary in different possible worlds of our model. Also let '9' be a constant symbol interpreted as the number 9 at every possible world. (This assumes that numbers are in the domain of our model, of course.) Now what about the assertion, 'necessarily the number of the planets is nine'? If we read it as $\langle\lambda x, y.\Box(x = y)\rangle(n, 9)$ it is true – the number of the planets is, in fact, 9, and 9 is 9 no matter what. But if we read it as $\Box\langle\lambda x, y.x = y\rangle(n, 9)$, it is quite different. This amounts to asserting synonymy, and is false.

Or again, say 'm' and 'e' are intended to denote the morning and evening stars respectively – in the actual world they denote the same object, but in other situations they need not do so. In the actual world, $m = e$ is the case, and hence $\langle\lambda x, y.\Box(x = y)\rangle(m, e)$ is true. But $\Box\langle\lambda x, y.x = y\rangle(m, e)$ is not so. In words, it is true, of the morning star and of the evening star, that they are identical, and this identity is necessary (as identity between objects is always necessary, if true). But it is not true that the morning star and the evening star are necessarily identical, that is, it is not true that the terms are synonymous.

7 Rigidity

In an example above I used a constant symbol, '9', which was interpreted to designate the same object in all possible worlds – the number 9. This is an example of a *rigid* term. For S_5, rigidity can be expressed quite simply: a term c is rigid in a model just in case the formula $\langle\lambda x.\Box(x = c)\rangle(c)$ is valid in the model. A little thought will make it clear it is asserting that, whatever c designates at a world, it designates the same thing at all worlds – in other words, its interpretation is a constant function.

Kripke and others have made the case that names in ordinary language are used rigidly (Kripke 1980). According to this theory, a name like 'Moses' received its initial designation at some point in the past and, by a complex process, some version of that designation has been passed down to us. This contrasts with definite descriptions. According to the Biblical account, Moses led the Israelites out of Egypt, but we can still make sense of a claim that he might not have done so. 'Moses' designates rigidly, but 'the person who led the Israelites out of Egypt' does not. Definite descriptions will be discussed later in this chapter.

8 De Re/De Dicto

Suppose we say 'The British monarch is necessarily the head of the British government.' This can be read in two different ways. On the one hand, we might be asserting the necessity of a particular statement, 'the British monarch is the head of the British government.' In this case, the necessity operator is used in a *de dicto* way, applying to a sentence (dictum). On the other hand, we might be ascribing a certain necessary property, 'necessarily being the head of the British government,' to an object, in this case the person who happens to be the British monarch. Such a usage of necessity is *de re*, ascribing a necessary property to a thing (res). In the present example, the *de dicto* version is correct, since the British monarch is defined to be the formal head of the British government. But the *de re* version is not correct since a British monarch could abdicate, and so no longer be government head.

To formalize the notions of the previous paragraph, suppose we introduce a constant symbol 'm,' intended to designate the 'British monarch' individual concept. That is, at each possible world it designates whoever is British monarch under those circumstances. And suppose we introduce a one-place relation symbol 'H,' intended to designate the intensional notion of being the head of the British government. It is easy to see that the *de re* version formalizes as $\langle\lambda x.\Box H(x)\rangle(m)$, while the *de dicto* version becomes

$\Box\langle\lambda x.H(x)\rangle(c)$. These certainly look different, and one can easily produce models in which they are not equivalent.

It does sometimes happen that, for certain terms, *de re* and *de dicto* usages coincide. Let us say that *de re* and *de dicto* are equivalent for a constant symbol c, in a model, provided $\langle\lambda x.\Box\Phi\rangle(c)$ and $\Box\langle\lambda x.\Phi\rangle(c)$ are equivalent at every world of that model, for every formula Φ. The question is, when does such a event occur? And the answer is quite simple: *de re* and *de dicto* are equivalent for c in a model if and only if c is rigid in that model. In particular, *de re* and *de dicto* are equivalent for names, assuming the Kripke et al. thesis. A proof of all this is not difficult, but I omit it here – one can be found in Fitting and Mendelsohn (1998).

9 Partial Designation

I've been assuming that terms always designate, but this is simplistic. A name, for instance, takes on a designation at a certain time, and before that it designates nothing. Definite descriptions provide another example. ‘The present King of France’ does not designate, though there were times when it did. To treat such things the notion of model must be somewhat expanded.

From now on the definition of interpretation is modified. If $\langle\mathcal{G}, \mathcal{R}, \mathcal{D}\rangle$ is an extended frame, an *interpretation* $\mathcal{I}$ is a mapping that behaves as before on relation symbols, but that assigns to each constant symbol c a mapping from *some* set of possible worlds (not necessarily all of them) to the domain of the frame. If a possible world Γ is in the domain of $\mathcal{I}(c)$, I'll say c *designates* at Γ. The definition of $(v * \mathcal{I})$ must also be modified: if c does not designate at Γ, $(v * \mathcal{I})(c, \Gamma)$ is undefined, and otherwise things are as they were.

Of course the definition of truth in a model must be modified as well. Partial truth assignments might be introduced – a formula could be true, false, or lack a truth value, at a possible world. This is an interesting direction, but it is not what is done here. I will simply assume that any ascription of a property to a term that does not designate is false. Formally, item 1 of the definition of $\mathcal{M}, \Gamma \Vdash_v \Phi$ is replaced by the following. (Recall, atoms can involve relation symbols or predicate abstracts.)

1. For an atom $R(t_1, \ldots, t_n)$,
   (a) if any of $t_1, \ldots, t_n$ do not designate at Γ then $\mathcal{M}, \Gamma \nVdash_v R(t_1, \ldots, t_n)$,
   (b) if all of $t_1, \ldots, t_n$ designate at Γ then $\mathcal{M}, \Gamma, \Vdash_v R(t_1, \ldots, t_n)$ just in case $\langle(v * \mathcal{I})(t_1), \ldots, (v * \mathcal{I})(t_n)\rangle \in \mathcal{I}(R)(\Gamma)$.

The rest of the definition remains the same.

I are *not* assuming any formula involving a non-designating term is false – only atoms. Among atoms, one in particular stands out: $\langle\lambda x.x = x\rangle$. In a model, at a world, if c fails to designate, $\langle\lambda x.x = x\rangle(c)$ will be false. But if c does designate, $\langle\lambda x.x = x\rangle(c)$ obviously must be true. Thus this abstract can serve as a convenient ‘designation’ predicate, and we give it that official role: $\mathbf{D}$ abbreviates $\langle\lambda x.x = x\rangle$. Now, c designates at a world if and only if $\mathbf{D}(c)$ is true at that world. If c does not designate, $\neg\mathbf{D}(c)$ will be true. This illustrates what was said above: there are true sentences involving non-designating terms.

10 Designation and Existence

Recall that earlier an existence relation symbol, E, was introduced. Using it, a pair of interesting abstracts can be defined.

$\mathbf{E}$ abbreviates $\langle \lambda x.\mathsf{E}(x)\rangle$.
$\overline{\mathbf{E}}$ abbreviates $\langle \lambda x. \neg\ \mathsf{E}(x)\rangle$.

Strictly speaking, $\mathbf{E}$ behaves the same as E and so is not really needed, but having it makes a nice symmetry with $\overline{\mathbf{E}}$. At a possible world Γ, $\mathbf{E}(x)$ is true just when x has as value an individual object that exists at Γ. Likewise, at Γ, $\overline{\mathbf{E}}(x)$ is true just when x has as value an individual object that is in the domain of the model but not in the domain of Γ, in other words, an object having possible but not actual existence at Γ. Since our possibilist quantifiers range over the domain of the model, we have the validity of $(\forall x)[\mathbf{E}(x) \vee \overline{\mathbf{E}}(x)]$ – quantifiers range over what has actual or possible existence. We also have that $(\forall x)[\overline{\mathbf{E}}(x) \equiv \neg\mathbf{E}(x)]$ is valid.

Constants are a different story, since they have intensional objects as values, and such objects might be partial. If c does not designate at a possible world Γ, neither $\mathbf{E}(c)$ nor $\overline{\mathbf{E}}(c)$ will be true at Γ, by part 1a of the definition of truth. On the other hand, if c does designate at Γ, it must designate something that actually or possibly exists. Putting all this together, we have the validity of $\mathbf{D}(c) \equiv [\mathbf{E}(c) \vee \overline{\mathbf{E}}(c)]$ for constant symbols.

All this is a little reminiscent of Meinong (1889). Think of $\mathbf{D}(c)$ as analogous to asserting that *c has being*. If c has being, it might or might not actually exist. In this sense 'the golden mountain' has being, does not actually exist, but could. Where the present treatment diverges from that of Meinong is, strictly interpreted, 'the round square' cannot designate at any possible world since the conditions are contradictory, and hence we cannot even say it has being. This point is related to the fact that, while a pair of abstracts $\mathbf{E}$ and $\overline{\mathbf{E}}$ was introduced, there was no companion for $\mathbf{D}$. An abstract $\langle \lambda x. \neg\ (x = x)\rangle$ could be considered, of course. But, for every constant symbol c, $\langle \lambda x. \neg\ (x = x)\rangle(c)$ will always be false. If c does not designate, it is false because no abstract correctly applies to a nondesignating term. If c does designate, it is false because the object designated must be self-identical. Roughly speaking, non-being is a property, but an uninteresting one since it never correctly applies to any term.

Going a little further, suppose c does not designate at Γ. Then $\mathbf{E}(c)$ will not be true at Γ, so $\neg\mathbf{E}(c)$ will be true. Of course $\overline{\mathbf{E}}(c)$ will not be true since c does not designate. It follows that $[\overline{\mathbf{E}}(c) \equiv \neg\mathbf{E}(c)]$ does not hold. This looks like a clash with the validity of $(\forall x)[\overline{\mathbf{E}}(x) \equiv \neg\mathbf{E}(x)]$, but recall that quantifiers range over individual objects, while constant symbols represent intensional objects, and may fail to designate. In fact universal generalization, $(\forall x)\Phi \supset \langle \lambda x.\Phi\rangle(c)$, is not valid – it fails when c does not designate. What we have instead is the validity of $(\forall x)\Phi \supset [\mathbf{D}(c) \supset \langle \lambda x.\Phi\rangle(c)]$.

11 Definite Descriptions

Definite descriptions, such as 'the King of France,' can be translated away into the primitives of our language, or they can be treated as primitives themselves. I'll straddle the fence, so to speak, and present both approaches.

To treat them as primitives the language must be enlarged, so that if x is a variable and Φ is a formula, then $\iota x.\Phi$ is a term with free variables those of Φ, except for x. The term $\iota x.\Phi$ is read, 'the x such that Φ,' or more briefly, 'the Φ.' The expanded definition of the term uses formulas, but the definition of formula uses terms, so it no longer is the case that terms can be defined first, and then formulas – the two must be defined simultaneously. This complicates things, but the obvious mutually recursive definition works fine. I'll skip over the details.

Next, the definition of designation for terms must be extended to include them. Suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is a model and $\iota x.\Phi$ is a definite description. I'll say this definite description *designates* at possible world $\Gamma \in \mathcal{G}$ just in case there is *exactly one d* in the domain of the model such that $\mathcal{M}, \Gamma \Vdash_{v[x/d]} \Phi$. Take $\mathcal{I}(\iota x.\Phi)$ to be the mapping whose domain is the set of possible worlds at which $\iota x.\Phi$ designates, and assigns to a possible world Γ in its domain the unique d such that $\mathcal{M}, \Gamma \Vdash_{v[x/d]} \Phi$.

According to this definition, if $\iota x.\Phi$ does not designate at possible world Γ, then $\langle \lambda y.\Psi \rangle(\iota x.\Phi)$ is simply false at that world, for any formula Ψ. In particular, $\langle \lambda y.\Phi \rangle(\iota x.\Phi)$ will be false. On the other hand, if $\iota x.\Phi$ does designate at world Γ, it is immediate from the definition that $\langle \lambda y.\Phi \rangle(\iota x.\Phi)$ is true at Γ. We thus have the simple principle $\mathbf{D}(\iota x.\Phi) \equiv \langle \lambda y.\Phi \rangle(\iota x.\Phi)$. In the present world it is not true that 'the King of France is King of France,' because the definite description 'the King of France' does not designate.

Russell (1905) showed that definite descriptions could be translated away in context, essentially saying that while they have the appearance of terms, formulas containing them are really abbreviations for more complex constructions. Stating Russell's translation in present notation, $\langle \lambda y.\Psi \rangle(\iota x.\Phi)$ is taken as abbreviating the formula $(\exists z)\{(\forall w)[\langle \lambda x.\Phi \rangle(w) \equiv (w = z)] \wedge \langle \lambda y.\Psi \rangle(z)\}$. That is, we have a formula asserting exactly one object has the property $\langle \lambda x.\Phi \rangle$, and that object also has the property $\langle \lambda y.\Psi \rangle$. It is not hard to see that a Russell approach is equivalent to the approach taking definite descriptions as primitive. The same formulas are validated either way.

Ontological arguments provide interesting examples of definite descriptions at work. Let's begin with one in the Descartes style. Suppose we define God to be the necessarily existent being – take g to be short for $\iota x.\Box\mathbf{E}(x)$. A definite description has its defining property if and only if it designates, so we have the validity of $\mathbf{D}(g) \equiv \langle \lambda y.\Box\mathbf{E} \rangle(g)$. But in this case we can do better – we also have $\mathbf{D}(g) \equiv \Box\mathbf{E}(g)$. This is not because of general principles about definite descriptions, but because of the particular form involved, $\iota x.\Box\mathbf{E}(x)$, and the fact that the underlying logic is S_5. (Proof of validity takes some work – give it a try.)

Continuing: for any term c, $\mathbf{D}(c) \equiv [\mathbf{E}(c) \vee \overline{\mathbf{E}}(c)]$. It follows that $\mathbf{E}(g) \supset \mathbf{D}(g)$ is valid. Combining things, we have the validity of $\mathbf{E}(g) \supset \Box\mathbf{E}(g)$. From this, by standard modal logic manipulation, we get the validity of $\Diamond\mathbf{E}(g) \supset \Diamond\Box\mathbf{E}(g)$. Since our modal logic is S_5, $\Diamond\Box X \supset \Box X$, and so we have the validity of $\Diamond\mathbf{E}(g) \supset \Box\mathbf{E}(g)$. This is a crucial step in Descartes' argument: God's existence is necessary, if possible. To complete the proof, we must establish the validity of $\Diamond\mathbf{E}(g)$. Unfortunately, at this point Descartes simply assumed it to be the case. I'll leave it to you to verify that $\Diamond\mathbf{E}(g)$, $\Box\mathbf{E}(g)$, and $\mathbf{E}(g)$ all turn out to be equivalent, so the Descartes assumption really begs the question.

The first ontological argument, historically, was that of Anselm. I'll conclude this section with a very informal discussion of it. This time, define God to be the maximally conceivable being. That is, God is the being such that I can conceive of nothing greater. Now let g abbreviate the informal definite description, 'the maximally conceivable being.' We have $\mathbf{D}(g) \equiv [\mathbf{E}(g) \vee \overline{\mathbf{E}}(g)]$, so if we assume that g designates (in the actual world), we have either $\mathbf{E}(g)$ or $\overline{\mathbf{E}}(g)$. Making reasonable assumptions, if we had $\overline{\mathbf{E}}(g)$ we would have a contradiction, because I can conceive of an existing God, and this would be greater than a nonexisting God. Consequently we must have $\mathbf{E}(g)$. This part of the argument is very imprecise, but we don't need to make it sharper because it is clear that it all depends on the initial assumption that g designates, and that was never verified. In short, the Anselm argument makes a plausible case that 'the maximally conceivable being' cannot designate a nonexistent being, but it does not establish that it designates anything.

12 What Next?

I've sketched the semantics for a first-order modal logic, and showed how it could be used to elucidate several topics of interest to philosophers. But the logic was, by design, a limited modal logic. There were quantifiers over individual objects, and constant symbols for individual concepts. One can complete the set by adding quantifiers over individual concepts and constant symbols for individual objects (Fitting 2000a). One can then consider whether or not to simply identify individual objects with individual concepts that are rigid. Technical issues are one thing, philosophical implications another. But this is beyond what we do here.

Going still further, one can introduce higher-type notions. We already have intensional relations – we could allow quantification over them, then add relations of relations, quantify over them, and so on. The intensional/extensional split that we have already seen continues upward through all these levels, and things become quite complex. Gödel devised an ontological argument of genuine interest, but to study it formally requires some machinery of this sort (Fitting 2000b).

Of course the more complicated things get, the less immediate our intuitions. The modal logic presented here is complex enough for many purposes, yet simple enough for us to grasp informally. Further exploration can be left to the intrepid.

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