Part VIII

LOGICAL FOUNDATIONS OF SET THEORY AND MATHEMATICS

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Logic and Ontology: Numbers and Sets

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From the standpoint of philosophical logic, a great gulf separates elementary arithmetic, understood here as involving only the so-called adjectival use of numerals, from advanced arithmetic which features their substantival use, where the distinction turns on a point of grammar. Thus, '5 is odd' will count as a truth of advanced arithmetic, with the substantival expression '5' serving as a proper name that denotes a Platonic entity (never to be seen on land or sea). If advanced, Platonic arithmetic takes numerical sentences grammatically at face value, it is nominalistic, adjectival arithmetic that proves more grammatically devious, in subjecting numerical sentences to a reductive paraphrase via a detour through first-order predicate logic. Thus 'there are at least 2 (i.e. two) Fs' will be paraphrased as (1).

(1) $(\exists x)(\exists y)$ (Fx Fy & ~ (x = y))

For in (1) we are quantifying in Quine's jargon (via the existential quantifiers '∃x' and '∃y') only over Fs, for example dogs, and any putative reference to 2 as a Platonic object in our ontology is deftly conjured away, in accordance with his slogan 'Explication is elimination'. Because philosophy of mathematics today can only be described as being positively spooked by the neo-nominalist challenge – first implicitly posed in Benacerraf's 1973 "Mathematical Truth" (Benacerraf and Putnam 1983) and soon after, much more aggressively, implemented in Field (1980) – my own agenda can be expected never to stray very far from the specter of that challenge.

In the nominalistic vein of (1) we can even take the proto-equation 'Two and two make four' to say that if there are at least two Fs and at least two Gs (no F being a G), then there are at least four Hs (every F and G being an H). Typographically, distinguish now adjectival 'two' from substantival '2,' thereby being afforded the opportunity of registering '2 and 2 make 4' at face value as a truth of advanced arithmetic. One and the same unregimented English sentence, where '2' and 'two' are taken to be synonymous, is seen here to be ambiguous, needing to be disambiguated as between a nominalistic 'two' and a Platonistic '2.' Because the nominalistic version of the sentence can be displayed as a valid statement-form of first-order logic, Frege's program of reducing arithmetic to logic is thereby vindicated. But only in respect to elementary arithmetic (as herein defined). For the vernacular '5 (or five) is odd' remains irreducible to first-order logic.[1]

No accident surely that predicate logic is also styled as quantification theory, reminding us that in addition to the general quantifiers '∃' and '∀' we are free to recognize the following numerical quantifiers: 'there are at least (exactly) two (three, four . . .) Fs' where the identity predicate found in (1) is smuggled in as a constituent of these complex quantifiers. No more than an advertising trick of relabeling, these ostensibly new quantifiers are already available for free in standard first-order predicate logic with identity. Recalling one traditional definition of mathematics as the science of quantity, we trust that more than a mere verbal trick is involved in now undertaking to recycle Frege's logistic thesis by assimilating mathematics as so defined to predicate logic characterized as quantification theory.

1 Sher's Weak Logicism

Pursuing that suggestion, one may even dare to emulate Gila Sher (1991) by enriching standard first-order logic with such adjectival yet Cantorian quantifiers as 'there are uncountably many x,' thereby inviting all of Cantor's alephs into 'logic.' Not that any pretense of reducing those alephs to logic along properly Fregean lines can be expected here, seeing that they will be supplied outright by Zermelo-Fraenkel set theory, taken to be coeval with logic itself. Even so, a convincing, recognizably Fregean case can be made for allowing the Cantorian quantifiers (as embedded in an extended first-order logic) to be certified as logical constants, thereby recasting set theory itself in terms of a weak logicism. Thus Sher writes, "Frege construed the existential and universal quantifiers as second-level quantitative properties that hold (or do not hold) for a first-level property in their range due to the size of its extension" (Sher 1991: 10). We may then suppose that the second-level property expressed by the Cantorian quantifier 'there are aleph-50 x' will fail to hold of the property expressed by the first-level predicate 'x is a dog' if only because the suggestion of there being aleph-50 dogs can only strike standard set theory as being impossibly droll, smacking of a category-mistake. Not at all the sort of scenario one routinely envisages.

Moreover, let all of space–time be packed solid with dogs cheek by jowl; that will yield no more than aleph-zero of them. A disappointing result really, for our program. In the general case one wants to say that for any F (absent information to the contrary) it is an open question how many Fs there are; and with dogs being the very sort of thing paradigmatic of what the question 'how many?' addresses, all of the cardinal numbers, transfinite as well as finite, ought to be available to draw upon. That at any rate is a new slant on set theory, viewed precisely as the general theory of cardinality, that is activated by Benacerraf's challenge, under the aegis of which the nominalist is free to argue that ruling out aleph-50 dogs is tantamount to ruling out aleph-50 itself as a genuine cardinal. No genuine cardinal, then no chance of figuring as a logical constant in its capacity of being a genuine quantifier.

If Sher's program draws on A. Mostowski's seminal (1957) "On a Generalization of Quantifiers," the latter in its turn draws on Cantor's generalization of the concept of (finite) cardinality, both of which can only prove the more nominalistically attractive if aleph-50 dogs were to be admitted as a serious option. The immediate obstacle lay in the exiguous accommodations that all of (our) space–time affords to dogs, combined with the tacit insistence (reminiscent of Kant) that there can only be one Space. Long out of favor, Kant's synthetic *a priori* has recently been making a modest comeback (notably in van Cleve 1999 but see also Tennant 1996), and it may now be invoked in support of the One Space thesis. Nominalists, however, will be least inclined to defer to it, preferring to canvass the copious plurality of worlds of Lewis (1986) as well as Everett's 'many worlds' hypothesis in quantum physics. In rejecting any aprioristic constraint on the cardinality of dogs, one must recognize (according to the official semantics anyway) that simply to say 'There are aleph-50 dogs' is to say that the *set* of dogs is equinumerous (via one-to-one correspondence) with . . . Assume here the simplest case where the (generalized) Continuum Hypothesis is true. Then the net result of filling the gap – with the words 'the power-set of the power-set of . . . the natural numbers' – will be that even nominalists can endorse the following argument A as valid: 'there are aleph-50 dogs, therefore there are at least aleph-50 Platonic objects,' sticking with our standard semantics. Assume, however, with Field (1993) that the (non)existence of sets is a contingent matter, meaning that given any set of dogs those dogs can jointly exist in the absence of all sets. Pretend now that the premise of A is true, and focus on the very dogs themselves, ignoring their cardinality. Following Field, those dogs will be found in a possible world where there are no sets, and there will even be aleph-fifty dogs there, seeing that for each dog here there will be exactly one dog there (indeed the same dog).

Not substantival 'aleph-50' then but rather an adjectival (and nominalistic) 'aleph-fifty' emerges as a logical constant (and transfinite quantifier) in a Sherian extension of first-order logic that a Fieldian nominalist can accept. Further warrant for speculations in that vein will be found in the megethology of Lewis (1998: 203–29) where a nominalistic universe is envisaged with quite as many entities as ZF supplies, doubtless to be sought in his plurality of worlds.

2 Finiteness, an Infinite Sentence and Skolem

Recoiling from these excesses, one may well wish to stick with first-order logic plain and simple, though (in the absence of both set theory and substantival arithmetic) Shapiro (1991: 9) indicates how one will then lack even the mere means to say that there are only finitely many dogs, or worse still (since it threatens first-order logic itself) that every first-order sentence consists of only finitely many expressions, thereby in effect joining Field in his recent "doubts about the determinacy of the notion of finiteness" (Schirn 1998: 99). It is here above all, with the finite itself, that nominalistic doubts about numbers and sets trickle all the way down to logic. Encouraged, however, by at least two linguists (Langendoen and Postal 1984), who insist that the grammar and syntax of ordinary language allow for sentences of (any arbitrary finite or) transfinite length, one can always liberalize our standard first-order logic and (try to) say in an infinitary notation, "There are exactly one or two or three or . . . dogs," in the adjectival mode of (1).[2]

How our failure to complete the sentence (whose length is presumed to be $\omega + 5$) might bear on (the constraints of) logic, may prove (a little) less obscure in the light of a marvelous exchange in Shapiro (1991: 206) where in reply to "What I mean by 'natural number' is 'member of the sequence 0, 1, 2, 3 . . .'," a Skolemite skeptic queries the meaning (and use) here of the dots . . . , along the lines of the following, simpler scenario of mine. Posit an infinite sequence of men, S, with a first, second, . . . where the dots can be shown to harbor a genuine indeterminacy which – to our astonishment – Skolem reads back into the natural numbers themselves. Naively, the two sequences, that is N and S, are on a par but the second allows for a man in the (the?) infinite sequence who is separated from the first man by infinitely many intermediate men. Start with a shortest man 6 ft. tall launching an infinite progression with each man taller than his predecessor yet with none reaching 7 ft. Here then is one infinite sequence, S_1, whose ordinality is ω. Add to it a 7-ft man, yielding a second sequence, S_2, of order-type $\omega + 1$. Providing for S_1, do the dots in my scenario of S also extend to our last man in S_2? Although the answer is doubtless no when it comes to the speech-act pragmatics of most occasions where S_1 will supply our standard or intended model of S, I take Skolem to be saying that as to mere semantics '. . .' as it figures in my scenario is infected with indeterminacy as between S_1 and S_2.

Because the implicit exposure of most people to Peano's (five) postulates for arithmetic extends only to the first four, which allow for just the sort of nonstandard model as my scenario (hence the need for Peano's fifth postulate), it can be a great mystery how the vulgar ever do acquire the concept of a natural number. Another, more philosophically urgent puzzle turns on how *we* succeed in doing so, even as favored with the fifth postulate of mathematical induction that – according to Skolem's devious argument, as richly discussed both in Shapiro (1991) and in Lavine (1994) – also fails to secure determinacy for it. Reminiscent of Kripgenstein's 'quus' paradox with its skepticism (Kripke 1982) as to how '+' can signify addition rather than quaddition, Skolem's regarding '. . .' will probably be fully resolved only after Kripgenstein's much more general worries about meaning and reference have been appeased, not to mention Putnam's (1977) "Skolemization of absolutely everything" (Putnam 1983: 15). If the most intriguing response to Kripgenstein lies in the 'saving constraint' of objective similarity out in the world that is "not of our own making," invoked in Lewis (1999: 45–55, 63–7) in order to fix content, one can at least see how it bears on Skolem, for '. . .' is doubtless synonymous with 'etc.' which in its turn, just means 'and (all) the others (of the same sort)' where what is to count as the same sort of like items needs to be pinned down. No surprise surely if Tennant's Schema C in the next section, which undertakes to ground the natural numbers in logic itself, should come to supply a piece in the puzzle.

3 Back to Strong Logicism?

Highly controversial, Crispin Wright's (1983) reactivation of Frege's logistic program, which for decades just about everyone assumed to be a lost cause, has forced researchers to rethink some of the more fundamental issues in logic. Benacerraf himself in a retrospective look at responses to his 1973 challenge regards Wright's 1983 as "the only line of inquiry that seems at all sensitive to arithmetical practice" (in Schirn 1998: 57). Neatly sidestepping Russell's Paradox of 1902 which ditched Frege's mature program, Wright reverts to an earlier version that features HP (Hume's

Principle) where the functional expression 'the number of x such that x is F' is abbreviated by 'Nx : Fx'.

(HP) Nx : Fx = Nx : Gx iff the Fs can be placed in one-to-one correspondence with the Gs.

Better still, Tennant (1997a: 310) features the simplified Schema C which directly effects an a priori synthesis of adjectival with substantival (finite) arithmetic.

(C) There are n Fs iff Nx : Fx = *n*

where '*n*' on the right indicates a numeral and 'n' on the left indicates its adjectival correlate which is to be unpacked as in (1). Taking HP and C to be analytic propositions, it will be easy now with either to produce an a priori proof of (the existence of) the natural numbers. Thus using C to derive '~(∃x) ~ (x = x) iff Nx:~(x = x) = 0,' we find that 0 emerges as the number of things which fail to be identical with themselves, while 1 emerges as Nx:(x = 0) and 2 as Nx:(x = 0 ∨ x = 1), etc.

Being analytic, won't all of these propositions, for example '(∃x) (x = 9),' be on a par with 'All bachelors are unmarried' and hence merely verbal truths that can tell us nothing about the world, being true solely by convention? Prompted by early Wittgenstein and dominant in the 1930s, the Conventionalist doctrine of analyticity, which was widely used to deflate Frege's program, has long been absent from contemporary discussions. Nor have I seen any sign of its being revived in response to Wright, largely (I suppose) thanks to Tarskian semantics, with an assist from Davidson (1967a), for whom to grasp the meaning of a sentence is to grasp its truth condition in terms of a Tarskian biconditional. Thus to grasp the meaning of the sentence 'All unmarried men are unmarried' is just to recognize that the sentence is true iff all unmarried men are unmarried, where our attention, initially fixed on a sentence, is guided away from it to the world, maybe even to the state of affairs of each unmarried man's being unmarried, though the genius of Tarski lay in declining to reify states of affairs. The mere availability of that option, however, has been widely felt to dispel Conventionalism. More characteristic of current controversy over neologicism – ranged on one side are Wright, Hale (1988) and Tennant, on the other Dummett (1991), Boolos (1998) and Field – is Field (writing in 1984), "I don't see how the existence of objects of any sort [e.g. numbers] can follow logically from the existence of objects of an entirely different sort [e.g. planets]" (Field 1989: 166) as in 'There are nine planets. ∴ The number of planets is 9' where our typographical innovation alerts us to a difficulty that goes unnoticed in the vernacular when premise and conclusion are seen as broadly synonymous.

Despite being widely though perhaps only subliminally shared, Field's worry detracted very little from the acclaim Davidson enjoyed when in "The Logical Form of Action Sentences" (1967b), in a comparable case, he urged that the following should count as a valid argument, to be certified as such by first-order logic: 'Tom is walking slowly. ∴ (∃x) x is a walking & x is slow' (Davidson 1982: 105–48) even though the premise was standardly supposed, by Quine and Co., to involve 'ontological commitment' only to Tom, while the conclusion was widely held at the time to feature a very dubious sort of entity, namely an event. Assume, however, that real logic is just first-order logic (a view still very much in fashion), and try formalizing the valid argument 'Tom is walking slowly. ∴ Tom is walking' (where the adverb proves recalcitrant) without positing in the premise (the event of) Tom's walking.

Still another example of Frege's legerdemain of a priori synthesis, which cannot then fail to smack of Kant's synthetic a priori, is found in Armstrong's aprioristic appeal to the principal Truthmaker – every true statement is made true by some object(s) – whereby the argument 'Socrates is wise. ∴ There is at least one state of affairs (of Socrates being wise)' emerges as valid (Armstrong 1997: 115). Frege, Davidson, Armstrong: these eminent instances of (what one might pejoratively call) philosophical logic invite Field to reply, "Plain, flat-footed logic is good enough for me." If in connection with the former sort of logic my harking back to the synthetic a priori will strike some readers as quaintly anachronistic, two considerations may appease them. First, the more general. Frege invents modern logic for one purpose only, namely to prove in detail the analyticity of arithmetic, as against Kant's synthetic a priori construal of it that may now be feared to infect HP and Schema C in a return of the repressed. Even assuming their analyticity, however, Tennant (1997b: 293–4) rightly senses a difficulty in Gödel's true but unprovable sentence of arithmetic that might then be urged to have a synthetic a priori status by way of contrast.

My second consideration bears directly on HP which Hale (1994: 124) urges us to view in terms of "a broader conception of analyticity that covers such cases" as 'Nothing is both red and green all over,' a proposition much contested over the years that van Cleve (1999: 226–9) joins a distinguished tradition in defending as paradigmatic of the synthetic *a priori*. Resolving this dispute between Hale and van Cleve offers the best researchable prospects today for assessing neologicism.

4 Benacerraf's Challenge

By invoking Davidson's events and Armstrong's states of affairs – both taken to be concrete entities – as foils for understanding Frege's abstract objects, we are given a window of opportunity for meeting Benacerraf's challenge. Recall the truffle in Hermione's hand that causes her to believe in its existence, catering thereby to the recent shift in epistemology, away from the traditional, internalist emphasis on evidentialism (the weighing of evidence) and toward the new externalist focus on reliabilism, with underlying belief-forming mechanisms that track the truth. Platonic objects in their causal impotence prove thus to be at a distinct naturalistic disadvantage when compared with the true-belief-inducing efficacy of perceived truffles. An unfair contrast when it comes to playing the ontology game! Contrast rather Frege in his ontology quantifying over 0 with Armstrong quantifying in his over the state of affairs of a truffle in his hand being seen by him, altogether waiving the success or failure of either's philosophical line of argument. Although Armstrong's everyday true belief in the truffle is caused by it, no such reliable mechanism is naturalistically credible when he undertakes to posit in his armchair ontology such *recherché* entities as his universals and states of affairs. Think here of Hume: the belief in truffles is caused by force of nature or habit. Not so with more rarefied speculations where ontologists go different ways. Let Armstrong's universals be allowed to exist. No matter. They play no role, externalist or internalist, in explaining why Armstrong does and Quine does not believe in them.

Actually, truffles themselves lose their innocence on being co-opted into (or banished from) the ontology game. After eliminating all Platonic objects, our nominalist may become emboldened to wield Ockham's razor afresh, now eliminating even truffles in the course of quantifying only over the elementary particles of physics. So our belief in truffles may not really be caused by them, but by elementary particles suitably arranged (to mimic truffles)? If one recoils from this suggestion mereology may be invited to kick in, on the ground that "mereological wholes" like truffles, "are not ontologically additional to all their [ultimate] parts," namely the particles, being rather "*identical* with all their parts taken together" (emphasis in original). Given the particles then, the truffles are supplied by way of an "ontological free lunch" which "like other such lunches . . . gives and takes away at the same time. You get the supervenient for free, but you do not really get an extra entity" (Armstrong 1997: 11–12). No extra entity? So (∀x) (x is an elementary particle)? Even though (∃y) (y is a truffle) & ~ (∃z) (z is a truffle & z is a particle)? That Armstrong's legerdemain here may be all of apiece with Frege's, one is encouraged to believe when Tennant, affecting equal facetiousness, appeals to his Schema C by way of "getting something for nothing" (Tennant 1997b: 322), and the comparison between the two cases may even dispel the invidious bugbear of Platonism that attaches to one of them. Not that an alternative model for understanding neologicism cannot be found in Armstrong's case for states of affairs, seeing that his independent appeal to Truthmaker – arguably playing the role of Schema C – is not taken by him to involve any ontological free lunch.

5 An Anti-realist Frege?

If sets have been seen as being constituted by their members (Parsons 1983: 217, 275 and 286), truffles have been taken to be constituted by elementary particles, and in both cases the 'constitution' relation may be viewed either in a realist mode as being metaphysically deep or in an anti-realist one as smacking of eliminativism. In this scheme the causal impotence of the one sort of item and the (putative) causal efficacy of the other may come to play very little role in any final reckoning.

In a somewhat different vein Dummett adjudges Wright's neologicism to be a success but only if it is viewed in terms of an anti-realist Frege of 1884 in *Grundlagen*. Not to be confused with the realist Frege of 1893 in *Grundgesetze* who profits by his interim discovery of the sense/reference distinction. The shift turns on the difference between a thin (anti-realist) and a thick (realist) notion of reference where the former is content to view "any legitimate question about the meaning of a term, that is, about what we should call its reference [as being] reducible to a question about the truth or otherwise of some sentence in the language" (Dummett 1991: 192). Thus 'the number of planets' will (trivially) succeed in denoting an object if 'the number of planets = the number of Jones's fingers' is true, while (the really important point) the non-trivial truth-condition of the sentence will be reductively satisfied just in case there are (exactly) nine planets and fingers. As of 1894, however, compositionality will decree that a whole sentence can have a sense (and reference) only if each of its (unitary) parts antecedently has a sense (and reference), as when ‘that truffle’ embedded in ‘that truffle is white’ comes to enjoy thick reference thanks to Benacerref’s causal link.

Faithful then to the earlier anti-realist Frege, Wright’s neologicism is deemed by Dummett to fail in its more ambitious goal of achieving thick reference for numerals. What Dummett forgets to mention, however, is his amazing discussion (Dummett 1973: 503) where a Kantian Frege is envisaged even as to 1893, for whom arguably all reference proves in the end to be thin.

> Our ability to discriminate, within reality, objects of any particular kind results from our having learned to use expressions, names or general terms, with which are associated a criterion of identity. . . . [W]e can in principle, conceive of a language containing names and general terms with which significantly different criteria of identity were associated, and the speakers of such a language would view the world as falling apart into discrete objects in a different way from ourselves. . . . [F]or Frege . . . it is we who . . . impose a structure on [the world]. (Dummett 1973: 503)

How to square this Fregean anti-realism with our radical Ockhamist who, in refusing to quantify over macro objects, allows only thin reference to ‘that truffle’ as it figures in the true sentence ‘That truffle is white,’ should not be very difficult. One has only to view the radical Ockhamist as clearing the ground for the still more radical position of Putnam’s conceptual relativism (Putnam 1988: 113–16).

6 Second-order Logic and Sets

If an anti-realist neologicism affords one option, a realist version emerged in my proposal to assimilate Tennant’s Schema C to Armstrong’s Truthmaker as, in the one case, numbers and in the other case states of affairs are admitted to a realist ontology. As to sets in particular, it is so-called second-order logic, which Quine famously distinguishes from logic proper, that ostensibly treats ‘x ε y’ as a(nother) logical predicate along with ‘x = y’. Take the second-order sentence ‘(∃F) (Socrates is F)’ which is routinely read either as ‘there is a property Fness that Socrates has’ or as ‘there is a set of which Socrates is a member.’ Frege, however, gives it a special twist: ‘There is something, for example wise, that Socrates is’ as in ‘Wise is something that Socrates is which I am not,’ indicating how the first-order ‘(∃x) (∃y) (x = y)’ supplies only one way of disambiguating ‘Something is something,’ for there is also the second-order reading, ‘Something (e.g. Socrates) is something (e.g. wise)’ where a first-order ‘something’ is followed by a second-order ‘something.’ ‘Wise is something’ is thus complete on one reading but incomplete on the other when ‘Wise is something (that Socrates is)’ fails to yield a complete sentence (as truncated). Hence Frege’s *outré* incomplete entities or quasi-entities (his concepts and functions) none of which can be given a proper name.[3]

Psychologically at least, one can understand how the conceptual jump in Schema C from non-objectual (adjectival) arithmetic to the objectual (substantival) variety might cease to strike Frege as being objectionably abrupt only after he came to finesse the passage by positing the *tertium quid* of his (arguably) Platonistic quasi-entities; and I can well imagine efforts today along this line to narrow if not perhaps quite close the gap, for example Hodes (1990: 255). I leave it as an open question whether accepting second-order logic in terms of Frege's second-order 'something' should persuade one to regard Quine's criterion of ontological commitment, enshrined in our first-order 'something,' as being unduly restrictive. In effect, then, I am asking the nominalist if he can live with Fregean second-order logic, being quite prepared to find that nominalists may divide on this issue owing to an inherent indeterminacy that infects our notion of ontological commitment when it comes to Fregean concepts and functions.

More encouraging may then be felt to be mereology whereby '(∃F) Socrates is F' receives this fourth, expressly nominalistic reading, 'There is something (a mereological whole) of which Socrates is a part.' Virtually patented in Boolos (1984, 1985), the new device of plural quantification supplies a fifth, nominalistic reading, 'There are some things such that Socrates is one of them' where these last two readings (I verily believe) echo in a deep way Frege's second-order 'something.' How thanks to plural quantification (2) can clarify in particular P(N), that is the problematic power-set of the natural numbers, will emerge in the sequel.

(2) There are some natural numbers that no set has as its only members.

Mystifying? Not (a trivial case) if one's set theory allows only finite sets. The trick will be to see how (some) constructivists might entertain the truth of (2) even as regards the infinite subsets of N where at least nominally P(N) can still exist with all of the subsets of N, that is all which are left over after (2) has kicked in. Because not only Sher (1991) but also Shapiro (1991) is engaged in investing set theory with a logical or quasi-logical status, the one in a first-order framework, the other in a second-order framework, the familiar reservations of constructivists pose a threat to both programs that (2) can hardly fail to illuminate.

Not to be confused with Quine, however, who convicts second-order logic of just being "set theory in disguise," Shapiro sharply distinguishes his own, logical conception of set from the standard, non-logical, iterative conception of ZF where only the former is to be equated with second-order logic. Always relative to some restricted universe of discourse, for example N, the first-order variables of Shapiro will range over the objects in the domain, while the second-order variables will range over their various subsets, reminding us of Tarski's model theory. If set theory proper is dizzyingly vertical, the logical conception of set is seen to be manageably horizontal, with a set of all non-Fs, for example non-dogs, being allowed only in the latter, Boolean system which smacks more of Aristotle than of Frege. Why the manageable system cannot really be insulated from the high tides of ZF, becomes clear when one asks after the cardinality of P(N) now that the Continuum Hypothesis has been shown, by Paul J. Cohen, to be neither provable nor disprovable in ZFC. Arguably more subversive than even the constructivist challenge to classical mathematics, there is a widespread failure of nerve today as to whether C.H. can be said to have a determinate truth value at all, in defiance of Excluded Middle that insists on it.

As to how this failure of nerve might threaten Shapiro's logical conception of set, a cheeky proposal by Field I find to be especially suggestive. Cheeky, I say, because as a nominalist he might be expected to relish the spectacle of set theory imploding from within; and one can only query his good faith here in offering guidance to Platonists trying to cope with (just about) any Cantorian aleph being eligible, as a point of mere logic, to fix the cardinality of P(N). Elaborating on the distinction between constructible (rule-governed) and non-constructible (random) sets, Field would have us recognize the term 'set' as systematically ambiguous, depending on whether we are referring to $sets_0$, $sets_1$, $sets_2$ . . . all the way up the hierarchy of ZF ordinals (Field 1994). When it comes to $sets_0$ then C.H. will be true and the cardinality of P(N) will be aleph-one. In all other cases C.H. will be false, and the cardinality of P(N) will be, say, aleph-46 when it comes to $sets_{45}$.

If one's first reaction to Field's proposal will probably view it as being bent on trashing set theory, an instructive byproduct of his suggestion is that (2) taken now as a schema emerges as true whenever the vague term 'set' is replaced by any one of his subscripted precisifications of it. And Field aside, an irenic constructivist who recognizes that (thanks to Russell's Paradox) there are already some things that no set has its only members might allow in the same vein for there being some numbers, for example 8, 86, 862, 8625, . . . that by corresponding to the decimal expansion of a putative irrational number which no (finite) rule can generate, that is, 8625 . . . , are thereby disqualified from being the sole members of any set. No less accommodating, a moderate classicist might now be prepared to retreat from his espousal of non-constructible sets to a backup position supplied by (2) if only to stake out common ground in what has otherwise been a longstanding deadlock. The key point, of course, is that what really matters here is that some numbers do exist (or might, for all we know, exist) thanks to which (2) is or might be true, and that the putative existence of a set that gathers them up (or does whatever sets are supposed to do in respect of their members) can only play a secondary role. Because this suggestion as to 'what really matters' relies on Boolos's device of plural quantification, label it P.Q. for future reference.

Field's proposal takes on new importance when it is viewed in the light of Hallett's remarkable work (1984: 208) where, following Paul J. Cohen himself, he writes, "[W]e have no positive reason to assume that even only one application of the power-set axiom to an infinite set will not exhaust the whole universe," which is as much as to say that P(N) may not be a ZF set at all but rather a proper class of von Neumann. What *may* be the case for Cohen, Field in effect takes really to be the case, and thereby daringly proposes by means of philosophy alone to answer what is still standardly taken to be an open question in mathematics proper, namely 'what is the cardinality of P(N)?' For all of Field's subscripted subsets of N will be seen by any true classicist to be proper *subsets* of N, and hence no aleph will suffice to fix how "incredibly rich" (Cohen) is P(N), precisely the "point of view" here which Cohen as of 1966 "feels may eventually come to be accepted."

7 Skolem (Again) and Megethology

"The proponents of second-order logic *as logic*", Shapiro writes by way of exorcising Skolem (1991: 204), "hold that second-order terminology . . . is sufficiently clear, intuitive, or unproblematic. . . . The claim is that once a domain (for the first-order variables) is fixed, there is a reasonably clear and unambiguous understanding of such locutions as . . . 'all subsets' thereof." Unambiguous? Field we find, in coping with P(N) and C.H., denies that, with his radical disambiguation of the term 'set.' But let that pass and focus on our "reasonably clear understanding" of the locution 'all subsets of N.' Suppose that Jones (standing in for the constructivist) insists on each set's having only finitely many members. Does he mean something different from the rest of us by the expression 'all the subsets of the natural numbers'? Although many will be strongly inclined to say yes, I doubt if they are fully registering my query as it is intended, with heavy emphasis on the whole notion of meaning in analytical philosophy. For we can hardly expect to bring Jones into line by saying, "So you're thinking of (river) banks while we're talking about (money) banks," seeing that – to pursue the analogy – he denies that the term 'bank' as we use it, that is 'set' or 'non-constructible set,' can have a nonempty extension.

Why not then suspend judgment as to consructivist demands, allowing that (2) might well be true owing to P.Q.? For we can continue to believe in the existence of (whatever should turn out to be) *all* the subsets of N. No longer indeed remaining faithful to Shapiro's "working realism" with its injunction to take classical mathematics at face value, and there may now even be reason to fear being led down a slippery slope to Skolemism where what counts as 'all the subsets of N' is always relative to this or that frame of reference. Recall that Peano's fifth postulate of mathematical induction was presciently designed (really by Dedekind) to rule out Skolem's nonstandard models of N precisely by quantifying over *all the subsets of N*. So Skolem may then be vindicated in either one of two ways: (a) Field's insistence on disambiguating 'set' prompted by C.H. and (b) worries about (2) having to do with constructivist scruples.[4]

Not for the first time do I now wish to show how, here in connection with our uneasiness over (2), nominalistic considerations can be pressed into the service of classical set theory itself. Returning then to mereology but not the free-lunch version of Armstrong that undertakes to co-opt 'x is a part of y' as a topic-neutral logical constant, I rely rather on a very topic-specific version of it that, in drawing on the familiar distinction between things and the stuff of which they are composed, restores mereology to its proper home, namely Helen Cartwright's quantities. Pour some water into the Atlantic ocean. Fifty years later someone might scoop it up from the Pacific. In the interim that very (quantity of) water will be sloshing about in a very scattered form. Pretend that this 'water' is composed of Democritean atoms made of adamant where, in a world with a simple infinity of them, there will be a distinct quantity of adamant corresponding to each classical subset (which in turn corresponds to each mereological sum) of those atoms. Mereology and (quantities of) stuff thus appear to be tailor-made for each other, which explains why the negation of (3), which has been designed as a mereological counterpart of (2), can strike us as an analytic proposition, featuring both nominalistic devices, namely plural quantifications as well as mereology, that are systematically combined to yield the megethology of Lewis's thesis "Mathematics is megethology" (Lewis 1993: 3–23).

(3) There are some Democritean atoms the adamant of which fails to comprise some adamant.

Compare: there are some wooden houses the wood of which (they are composed) fails to comprise some (quantity of) wood. Here if anywhere mereology supplies an ontological free lunch, and P(N) can thus be envisaged in an entirely nominalistic realization yet free of all constructivist restrictions, thereby running afoul of Feferman (1998) that takes his constructivist program to be mandated by his rejection of set-theoretical Platonism. To the contrary, nominalists are free to divide, some opting for a classical, non-constructivist mereology, while others insist on a restrictive, constructivist abridgment of it. Platonism proves then to be largely a red herring when it comes to the key issue. How the debate (over constructivist vs. non-constructivist mereology) will play out in the coming decade, one can only await in suspense.

Because my Democritean scenario takes us beyond the actual world (recalling in this respect my conceit as to aleph-50 dogs), it does look as if my gropings in this chapter toward a replay of (something like) Frege's logicism will have to look to Hellman (1989) and Chihara (1990) for expressly modal foundations of mathematics. To the question whether the modal operators 'it is possible (necessary) that' should count as logical constants, may then be added queries as to 'x is a part of y,' '$x \in y$,' 'x is one of the ys,' and even '$x = y$.' How logic itself is finally to be characterized, can be expected to accommodate various considerations raised in this chapter, having to do with numbers and sets.

Notes

1 Because 5 is defined in the Frege–Russell program as the set of all sets with exactly five members, a reduction of '5' to 'five' is effected but only via the Platonic objects of set theory. After Russell's Paradox ZF set theory rules out any such set as being 'too large.' Acceptable as a proper class, 5 in that role is debarred from being a member of any set or class. No set or class then of natural numbers, spelling defeat of the program.

2 Responding in 1931 to Gödel's true but (finitely) unprovable sentence, "Zermelo went on to propose a massive infinitary logic" along this line, writes Shapiro (1991: 191), where "Zermelo argued that Gödel's reasoning shows that any finitary notion of 'proof' is inadequate." Zermelo's option remains problematically open to this day.

3 Although Dummett (1973: 243) concludes his ch. 7 on incomplete expressions by finding Frege's position to be "in the end unjustified," in his later book (1981: 164), he reopens the issue where "this conclusion now seems to be too strong."

4 In an especially perspicuous defense of Skolemism (as against Benacerraf), Wright remarks that "there are, I suppose, two routes into the informal notion of a subset of a given set. . . . Neither of these routes, it seems to me, holds any very plausible promise of meeting the Cantorian's needs" (Benacerraf and Wright 1985: 135). If Wright now has grounds for resisting (2), I have yet to learn of them, though I would not be surprised to meet a Skolemite who felt threatened by (2).

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