3

The Rise of Modern Logic

ROLF GEORGE AND JAMES VAN EVRA

The history of some sciences can be represented as a single progression, with each dominant theory coming to the fore, then eventually falling, replaced by another in succession through the centuries. The development of physics, for instance, can be understood as such a chain, connecting Newton in the seventeenth century with Einstein in the twentieth. Logic did not progress in this way; no dominant theory commanded it (a tapestry more than a chain) until the first decades of the twentieth century. No self-sustaining internal theory held sway before then, nor was there much rigor externally imposed. Even Aristotle, as one commentator put it, was more venerated than read, and most versions of syllogistic logic proposed after the Middle Ages did not measure up to the sophistication of his own system.

1 The Dark Ages of Logic

In 1543 the French humanist and logician Peter Ramus (1515–72), who had made a name for himself with his dissertation *Whatever Aristotle Has Said is False*, published his *Dialectic*, a slim book that went through 262 editions in several countries and became a model for many other textbooks. Ramus gratified the taste of the times by writing an elegant Latin, drawing his examples from Cicero and other classical authors, and by neglecting most of the finer points of medieval logic and the associated 'barbarous' technical vocabulary. The book was committed not to logic as we now know it, but to the art of exposition and disputation. Its first sentence, in an early English translation, reads "Dialecticke otherwise called Logicke, is an arte which teachethe to dispute well." In the next centuries, logic as the art of rhetoric and disputation, became the domain of textbook writers and schoolteachers, a prerequisite for careers in law or the church. The major authors of modern philosophy and literature did not advance or even concern themselves with logic so conceived, and generally treated it with derision. John Milton thought it a subject in which "young Novices . . . [are] mockt and deluded . . . with ragged Notions and Babblements, while they expected worthy and delightful knowledge" (*On Education*).

This was an age also of discovery in the sciences and mathematics. The textbook logic 'of the schools' played no role in this. Francis Bacon claimed in the *Novum*

Organum that the "logic we now have" does not help us to *discover* new things, but "has done more to . . . fasten errors upon us, than to open the way to truth" (Book 1, Aphorism xii). He advocated instead rules of induction, a methodology of scientific investigation. In the *Discourse on Method* Descartes made similar remarks and John Locke, more radically, thought unaided natural reason to be more powerful than any logical methodology:

> Native rustic reason . . . is likelier to open a way to, and add to the common stock of mankind, rather than any scholastic proceeding. . . . For beaten tracks lead this sort of cattle . . . not where we ought to go, but where we have been. (*Essay Concerning Human Understanding*, 4.17.7)

The "cattle," poor drudges who taught logic to undergraduates, struck back by proposing to ban Locke's *Essay* from Oxford, since "there was a great decay of logical exercises . . . which could not be attributed to anything so much as the new philosophy, which was too much read" (Cranston 1957: 465ff).

Hume continued Locke's attack: "Our scholastic headpieces shew no . . . superiority above the mere vulgar in their reason and ability" (*Treatise on Human Nature*, 1.3.15). Denis Diderot's article on logic in the *Encyclopédie*, the most widely consulted reference work of the century, claimed that reasoning is a *natural* ability; to conduct logical inquiries is like "setting oneself the task of dissecting the human leg in order to learn how to walk" (*Encyclopédie*, Logique).

Gottfried Wilhelm Leibniz was the great exception to the logic bashing of the seventeenth and eighteenth centuries. He saw the general outline of what logic would much later become, but left only fragments of a 'universal characteristic' through which it would be become possible, he thought, to settle philosophical disputes through calculation. In the *New Essays Concerning Human Understanding*, a dialogue in which he responded to Locke, the latter's representative Philateles eventually admits "I regarded [logic] as a scholar's diversion, but I now see that, in the way you understand it, it is like a universal mathematics" (*New Essays* 4.17.9).

Traditionally, an exposition of logic followed the sequence: theory of terms or concepts, their combination into judgments, and the composition of syllogisms from judgments. This was now commonly prefaced by a discussion of the origin of concepts, as inherent in the mind or deriving from sensation and perception. In the end, many logic books contained more of these epistemological preliminaries than logic. There was, further, especially in England, an ongoing emphasis on logic as the art of disputation.

2 Kant and Whately

For the disordered progress of logic to even get on a path that would lead to modern logic, a reorientation and elimination of materials had first to occur. Neither Kant nor Whately contributed substantially to the formal development of logic, but they played a major role in this eliminative exercise.

Kant, unaware of earlier and since forgotten progress in logic, held that logic did not have to set aside any part of Aristotle's theory, but also had not taken a single step forward, and "is to all appearances finished and complete" (*Critique of Pure Reason*, B viii). But in early lectures, he had shared the general disdain for the subject: "It took great effort to forget [Aristotle's] false propositions. . . . Locke's book *de intellectu* is the ground of all true *logica*" (Kant 1992: 16, 24).

By 1781, the time of the *Critique of Pure Reason*, he had changed his mind; Locke "speaks of the origin of concepts, but this really does not belong to logic" (Kant 1992: 439). While claiming earlier that the logician must know the human soul and cannot proceed without psychology, he now held that "pure logic derives nothing from psychology" (*Critique of Pure Reason* A54/B78).

Kant made two widely accepted distinctions: (1) he contrasted 'organon' and 'canon.' An *organon* (Kant uses the word in the sense Bacon gave it in the *Novum Organum*) attempts to codify methods of discovery. But "logic serves as a critique of the understanding, . . . not for creation." He sensibly held that there is no universal method of discovery, which rather requires a grasp of the special science that is to be advanced. But since logic must be general, attending only to form and not to content, it can only be a *canon*, a method of evaluation (*diiudicatio*). Methodological rules and theories of the origin and association of ideas, though intended as improvements of logic, are not even part of it. (2) Kant further divided logic into theoretical and practical. The latter, important but derivative, dealt with honing the skill of reasoning and disputation, while logic proper is a theoretical inquiry.

In the following decades nearly every German logic text was written by a student or follower of Kant. A contemporary could rightly observe that Kant gained a pervasive influence upon the history of logic. Regrettably, the overburden of psychology and epistemology in German logic treatises increased again in the course of the century, while its formal development stagnated, in part because of Kant's claim that it was a finished science.

Richard Whately (1787–1863) contributed to logic at the level of theory rather than formal detail. *Elements of Logic* (1827), an enormously popular response to the unrelenting criticism of the subject, was widely credited with reviving logic in England. Rather than fault logic for not doing what it cannot do (be an engine for discovery, or an "art of rightly employing the rational faculties"), it is better to focus on formal structures. In Whately's view, logic is an objective science like chemistry or mathematics, and its point (like that of the others) is the enunciation of principle apart from application. Faulting logic for not making people think better, "is as if one should object to the science of optics for not giving sight to the blind" (Whately 1827: 12).

Whately considered logic to be immediately about language, rather than vaguely conceived 'thought.' Unlike many of its loosely written predecessors, his book contains a formally adequate presentation of the categorical syllogism. A syllogism is a 'peculiar form of expression' into which any specific argument can be translated for testing validity. Properly understood, it is to an articulated argument as grammar is to language. The 'grammatical' analysis of any argument will lead to syllogistic form, just as the analytic devices of chemistry can be used on any compound and lead to basic elements. He also pushed an analogy with mathematics: just as the variables in mathematics stand for any number, so the letter variables used in stating syllogistic form stand for any term.

While Whately's theory is nearer to our present conception of logic, his critics faulted him for confining it within too narrow a scope. No longer would logic be the great sprawling subject that could be redefined almost at will, and many longed for that latitude. He prepared logic for innovation at the formal level.

3 Bernard Bolzano

At about the same time, Bernard Bolzano (1781–1848), "one of the greatest Logicians of all time" (Edmund Husserl), published his four-volume *Theory of Science* (*Wissenschaftslehre* (WL) 1837). It is the finest original contribution to logic since Aristotle, and a rich source for the history of the subject. In WL no formal calculus or system is developed; it is, rather, a treatise on the semantic concepts of logic. It was celebrated for its resolute avoidance of psychology in the development of these concepts.

Bolzano defines a spoken or written sentence as a speech act that is either true or false. Its *content*, that which is asserted or denied, is a proposition 'in itself,' explained as "any claim [*Aussage*] that something is or is not the case, regardless whether someone has put it into words, . . . or even has formulated it in thought" (WL § 19). He had little interest in the ontological status of these abstract propositions and meant to assert nothing deeper than we all do when we say that there *are* truths that are not yet known, or mathematical theorems not yet proved.

Any component of such a proposition not itself a proposition is a *Vorstellung* (idea or representation) in itself. The common sequence of first introducing terms or ideas and then propositions as compounds of them is here reversed. Bolzano noted that no one had successfully defined the type of combination of terms that generates a proposition. Several of the attempts he examined did not distinguish propositions from complex terms, 'the man is tall' from 'the tall man,' and others defined it in terms of 'acts of the mind,' contaminating logic with psychology (WL §§ 21–3).

Others (Hobbes, Condillac) identified propositions with equations, sometimes writing 'Caius is a man' as 'Caius = man.' Condillac and others maintained further that the principle on which all syllogisms rest is that two things equal to a third are equal to each other. But, Bolzano notes, while all equations are propositions, not all propositions are equations (WL §§ 23.20) and paid no further attention to this doctrine.

Identifying propositions with equations demanded further adjustments, the 'quantification of the predicate.' The German logician Ploucquet (1716–90) thought that in an affirmative proposition the predicate cannot be different from the subject. Hence he understood the proposition 'All lions are animals' as 'All lions are *some* animals.' In the same vein George Bentham (1800–84), in a commentary on Whately's book, symbolized 'All X are Y' as 'X *in toto* = Y *ex parte*' or 'All of X = Part of Y' (Bentham 1827: 133). The doctrine is now usually associated with the name of William Hamilton (1788–1856) who disingenuously claimed to have discovered it and gave it wide currency.

Back to Bolzano. He held that many propositions are not adequately expressed in common language. For instance, the proposition corresponding to the utterance

'I have a toothache' identifies speaker and time and is more adequately phrased as 'Neurath has a toothache at t.' Also, 'There is an A' is not, as it seems, about A's, but about the *idea* A; it means that this idea refers to an object (cf. Frege on quantifiers, below).

Bolzano's most important contribution was his definition of logical consequence using the mathematical technique of substitution on variables:

> Propositions M, N, O, . . . *follow from* propositions A, B, C, D, . . . with respect to the variable elements i, j, . . . if every set of ideas [*Vorstellungen*] whose substitution for i, j, . . . makes all of A, B, C, D, . . . true also makes M, N, O, . . . true. (WL § 155)

For example, 'a is larger than b, b is larger than c, therefore a is larger than c' is valid 'with respect to' the set of ideas 'a,' 'b,' 'c.'

It was generally understood, and often stated, that in a valid deductive argument, the conclusion follows *of necessity* from the premises (cf. Aristotle, *Prior Analytics* 24^b18). Bolzano's definition, closely akin to that given a century later by Alfred Tarski, was meant to explain the nature of this necessity.

If the variable elements i, j, . . . include *all* extralogical terms, then the consequence is said to be *logical*, as in a valid categorical syllogism. The unusual *triadic* construction of consequence also allows for enthymemes, or partly 'material' consequences, where only a *subset* of extralogical terms is varied. For example, in the argument 'All men are mortal, therefore Socrates is mortal,' any substitution on 'mortal' that makes the premise true makes the conclusion true: though not a logical consequence, it is valid with respect to 'mortal' (cf. George 1983).

Most logic texts of the period claimed, without supporting argument, that the so-called 'laws of thought' (identity, contradiction, and excluded middle) are the basic principles, the foundation on which all logic rests. While Bolzano agreed that these principles are true – his own logic was bivalent – his understanding of logical consequence showed him that nothing of interest followed from them. Logic, he maintained, *obeys* these laws, but they are not its *first principles* or, as we would now say, axioms (WL § 45).

He objected further to common attempts of grounding these laws in psychological necessities. Typically, the law of contradiction was supported by claims that a whole that is inconsistent cannot be united in a unity of thought, for example that round and quadrangular cannot be thought together because "one representation destroys the other." Against this Bolzano noted that we can, and often do, entertain inconsistent concepts. We can ask, for example, if there are regular dodecahedrons with hexagonal sides. But such a figure is just as impossible as a round square, only not obviously so. There are, in other words inconsistent ideas in themselves in Bolzano's abstract realm, and if entertained in a mind, they do not self-destruct.

Bolzano took mathematics to be a purely conceptual science, and disagreed with Kant's view that it was founded on intuition. Even in a diagram, what matters is what is general in it: the concept and not the intuition. His pioneering contributions to functional analysis entered the mainstream of mathematics in the nineteenth century, while his logical writings were appreciated only in the next.

4 John Stuart Mill

In his *System of Logic* (1843) Mill did not contribute to the development of logic as formal science, but like Bacon, attacked it. He claimed that formal principles, especially the syllogism, are a *petitio principii* since they can generate no new knowledge. One can know that the major premise 'All men are mortal' is true only if one knows the truth of the conclusion 'Socrates is mortal.' If that is still doubtful, the "same degree of uncertainty must hang over the premiss" (*System of Logic*, 2.3.2). When Archbishop Whately said that the object of reasoning is to "unfold the assertions wrapt up . . . in those with which we set out," Mill complained that he did not explain how a science like geometry can all be "wrapt up in a few definitions and axioms" (*System of Logic* 2.2.2). To explain that this is indeed the case had been a main objective of logic and mathematics before and especially after Mill. He thought it a project doomed to fail and claimed that the truths of geometry and arithmetic are empirically discovered by the simplest inductive method, that is *enumeration*. If a large number of instances of, and no exceptions to, A's being B is observed, it is concluded that *all* A's are B. Now if we have two pebbles and add another, then without exception we get three; neither do we ever observe two straight lines enclosing a space, forcing our minds to accept the truth of these and other mathematical propositions. Mill concluded that the "principles of number and geometry are duly and satisfactorily proved" by the inductive method of simple enumeration (*System of Logic* 3.21.2). Gottlob Frege later observed sarcastically that Mill never defined any number other than 3, nor did he illustrate the physical facts underlying 1 or 0, nor what "observed fact is asserted in the definition of the number 777846" (Frege 1884, § 7: 9).

Mill took the same empiricist and psychological approach to logic, whose "theoretic grounds are wholly borrowed from Psychology, and include as much of that science as is required to justify the rules of the [logical] art" (Mill 1865: 359). This holds in particular for the 'laws of thought,' which are grounded either in our psychological constitution, or in universal experience (1865: 381). Echoing earlier claims, he thought it impossible to entertain inconsistent concepts.

The *System of Logic* is best known for formulating rules for the discovery of causes, his famous 'canons': the methods of agreement, difference, residues, and concomitant variation. To illustrate the last: we take the moon to be the cause of tides, because the tides vary in phase with the position of the moon.

For a while, Mill's logic was the dominant text in logic and the philosophy of science in Britain, his eloquence creating much support to the view that logic is methodology and the art of discovery.

5 Boole, De Morgan, and Peirce

George Boole (1815–64) formulated his algebraic logic in conscious opposition to Mill's approach. Taking the mathematical analogy further than the loose suggestion of Whately, he sought to use algebra as a formal structure within which inferences could be perspicuously formulated. Logic should be a branch of mathematics, not of philosophy; this would excise methodology, rhetoric, and epistemology. But logic can be a branch of mathematics only if the latter is not construed, as was common, as the science of quantity, but as the science of symbolic operations in general.

In his *Mathematical Analysis of Logic* of 1847 Boole introduced the notion of an 'elective symbol,' for example 'x', which represents the result of 'electing' the x's from the universe; it is the symbol for the resulting class. xy is the result of electing y's from the class x, hence the intersection of the two classes. It holds that $xy = yx$ and also that $xx = x$. $x + y$ is the union of the two classes, $x - y$ elects the x's that are not y. 0 is the empty class and 1 'the universe,' hence $1 - x$ is the class of non-x's. It follows that $1x = x$, $0x = 0$ and $x(y \pm z) = xy \pm xz$. A universal affirmative, 'All x are y' becomes '$x(1 - y) = 0$,' which says that the class of things that are x and not-y is empty. While this is an equation, it should be noted that it does not identify the subject with the predicate, as we find in earlier attempts of introducing algebraic notation into logic. A proof of the syllogism *Barbara* illustrates the algebraic method:

The syllogism	*Boolean computation*	*Comment*
All M are P	1. $m(1 - p) = 0$	the intersection of m and non-p = 0
All S are M	2. $s(1 - m) = 0$	the intersection of s and non-m = 0
	3. $m = mp$	algebraically from 1.
	4. $s = sm$	algebraically from 2.
	5. $s = smp$	mp for m in 4, licensed by 3.
	6. $s = sp$	s for sm in 5, licensed by 4.
	7. $s - sp = 0$	algebraically from 6.
All S are P	8. $s(1 - p) = 0$	algebraically from 7. QED.

The conclusion follows by 'multiplying' and 'adding,' specifically by maneuvering the middle term into a position where it can be eliminated. Syllogistics becomes part of the algebra of classes and thus an area of mathematics. If every argument can be formulated as a syllogism, then all of logic is a part of algebra.

For every analogy there is some disanalogy, and Boole's link between logic and algebra (as he was fully aware) was no exception. Some arithmetic functions (such as division, and even some cases of addition and subtraction) did not easily admit of logical interpretation. There are also difficulties in Boole's rendition of existential propositions: he wrote 'Some X are Y' as $v = xy$ where v stands for a class whose only defining condition is that it not be empty. But how can one define such a class? Also, his logic was still a logic of terms. The recognition of even so elementary a sentential function as negation came only later in the century.

Augustus De Morgan (1806–71) took a different path, retaining a closer connection with traditional syllogistic logic but moving the subject far beyond its traditional limits. When stripped of unnecessary restrictions, the syllogism would constitute an adequate basis for the representation of all modes of deductive reasoning. In his *Formal Logic* (1847), and in a later series of articles, he pushed the syllogistic structure so far that he called the status of the standard copula – 'is' – into question. If that term could be replaced by any term relating the other components in the statement, the reach of the syllogism would be broadened: categorical statements would become relational statements.

De Morgan's more general interest in the logic of relations led him to examine inherently relational arguments, such as 'Every man is an animal. Therefore the head of a man is the head of an animal', which traditional syllogistic logic could not accommodate. He also introduced the concept of the 'universe of discourse,' still generally used, as a way of targeting statements to a class of objects under discussion, rather than the entire universe.

Charles Sanders Peirce's (1839–1914) theory of logic was once characterized as wider than anyone's. He was the first to consider himself not primarily a mathematician or philosopher, but a logician, filtering through the sieve of logic every topic he dealt with. On the formal level, he developed the logical lineage of Boole and De Morgan by refining the logic of relations, and devising more abstract systems of algebraic logic. He viewed it as a new and independent stage in the development of logic. The algebra of logic should be self-developed, and "arithmetic should spring out of logic instead of reverting to it." He developed a version of the modern quantifier, and of sentential functions. In both cases, it has been argued that, although Frege is often credited with introducing both notions into logic, it was Peirce and his students who were there first. Earlier he thought that logic is part of 'semiotics,' the theory of signs, their meaning and representation. Later he took it to *be* that theory, and while first taking logic to be descriptive, he later thought it to address cognitive norms.

Peirce introduced the memorable division of arguments into deduction, induction, and hypothesis, the last also called abduction and, more recently, 'inference to the best explanation.' He illustrated them as follows, using the then common terms 'Rule' for the major premise, 'Case' for the minor, and 'Result' for the conclusion of a categorical syllogism (Peirce 1931: 2.623):

Deduction:		*Rule*:	All the beans in this bag are white.
		Case:	These beans are from this bag.
	∴	*Result*:	These beans are white.
Induction:		*Case*:	These beans are from this bag.
		Result:	These beans are white.
	∴	*Rule*:	All the beans in this bag are white.
Hypothesis:		*Rule*:	All the beans in this bag are white.
	. . .	*Result*:	These beans are white.
	∴	*Case*:	These beans are from this bag.

In the last example the conclusion (the 'case') is accepted because on the available evidence it is the best explanation of why the beans are white.

6 Gottlob Frege

Frege (1848–1925) was a German mathematician and philosopher who set logic on a new path. He sought to connect logic and mathematics not by reducing logic to a form of algebra, but by deriving mathematics, specifically arithmetic, from the laws of logic.

He saw that a philosophy of language was a prerequisite for this and developed much of it in his *Conceptual Notation* (*Begriffsschrift*) of 1879. Like Bolzano, but more polemically, Frege opposed any attempt to import psychology into logic, repeatedly attacking Mill for this confusion. The meaning of sentences, for instance, is not explained by the mental states of speakers, but by investigating the language itself.

From the premise 'Castor is a sibling of Pollux,' two conclusions can be drawn by the *very same principle of inference*: 'Someone is a sibling of Castor' and 'Someone is a sibling of Pollux.' Traditionally, 'Castor' was construed as a different kind of sentential component than 'Pollux,' the first being the subject, the second lodged inside the predicate, so that the two conclusions followed by *different* principles. To correct this and other shortcomings of the traditional analysis of sentences, Frege replaced it with one built on *functions*.

In the equation √4 = |2| we distinguish *function* ('√'), *argument* ('4'), and *value* (|'2'|). The function is said to 'map' the argument to the value. '√( )' by itself is an 'unsaturated' expression that has a gap (shown as '( )') to be filled by an argument.

Frege construed sentences in the same way, '( ) is a planet' as a *sentential* function. If an argument, here called a *name* (an expression like 'Mercury,' 'Sirius' or 'the planet nearest the Sun') is inserted, a sentence results: 'Mercury is a planet' for example, or 'Sirius is a planet.' Sentential functions, like mathematical functions, can take more than one argument, as in '( ) is a sibling of { }', etc. In the Castor–Pollux example, the two arguments have the same status, and thus the single rule now called ∃-introduction, or existential generalization, legitimates both conclusions.

A function symbol refers to, or denotes, a *concept*, the name an *object*. Concepts and objects belong to distinct ontological categories. When a concept-term is an argument in a sentence, as in 'Red is a color,' the sentence is said to be on a 'higher level' than those whose arguments refer to objects.

As in the mathematical case, a sentential function maps its argument(s) to a value, but there are only two of these, the True and the False, the *truth values* of sentences. Thus the concept '( ) is a planet' maps 'Mercury' to Truth, 'Sirius' to Falsehood. In Frege's terms, Mercury 'falls under' the concept, Sirius does not. This is not just a more complicated way of saying that the one sentence is true, the other false. It is, rather, an analysis of what that *means*.

A further profound innovation was the quantifier. In mathematical texts quantification is usually tacit. For instance, 'x + 0 = x' is true if it holds for every integer. If sentential connectives are brought into play, this no longer works: 'Fx,' if taken in the sense of a mathematical formula, will mean that everything is F, and its denial '¬F(x)' that nothing is F, since it is true if ¬F(a) ¬F(b) etc. But 'Not everything is F' cannot be expressed in this way. For this, a special sign, a *quantifier* with a *scope* is needed. In current notation we can then distinguish between ¬∀F(x) and ∀x¬F(x). Frege took quantifiers to be higher level functions. The sentence 'There is a planet' is to be rendered as 'There is at least one thing such that [( ) is a planet].' The quantifier is here construed as a function that has another function as its argument.

Frege emphasized the importance of the 'deductive method.' Claims in a deductive science must be justified by a *proof*, which in his and all later logicians' view, is a sequence of propositions, each of which is either an assumption, or follows from previous members of the sequence by clearly articulated steps of deduction.

With this understanding of the structure of propositions, of quantification, and of the nature of a proof, *Begriffsschrift* develops an axiomatic system of sentential logic, based on two principles (actually two sets of axioms), one dealing with conditionals, the second with negation. The rule of *modus ponens* is employed to generate the first consistent and complete (as was shown much later) system of sentential logic.

A third principle, substitutivity, is introduced: if a = b, then F(a) is equivalent (as we now say) to F(b). With the introduction of a fourth principle, now 'universal instantiation' or ∀-elimination, a system of second order predicate logic is developed.

It seems that substitutivity fails in so-called *oblique* (or as we now say *opaque*) contexts. According to Frege, they are dependent clauses introduced by such words as 'to say,' 'to hear,' 'to believe,' 'to be convinced,' 'to conclude,' and the like. Now 'N believes that the morning star is a planet' may be true, while 'N believes that the evening star is a planet' false, even thought the two heavenly bodies are identical, apparently violating substitutivity. To save this principle, Frege introduced the important distinction between *sense* (*Sinn*) and *reference* (*Bedeutung*) (1892). 'The morning star' refers to the same object as 'The evening star' but they have a different sense. This is not the mental content associated with the signs, but their 'common meaning,' an objective entity determining the reference. Frege made the attractive assumption that in opaque contexts such expressions do not name an object, but their own sense, allowing substitution with any name of identical sense. Consider the sentence 'K believed that the evening star is a planet illuminated by the sun.' Here 'the evening star' may be replaced, *salva veritate* by 'the brightest star-like heavenly body in the evening sky,' provided the two expressions have the same sense for K. Similarly, sentences in oblique contexts have as their reference not their truth value, but the *thought* or sense they express. In this way, substitutivity, for Frege an incontrovertible principle of logic, can be made to work in opaque contexts.

Frege's main object was to show that arithmetic can be derived from logic alone, a project now called 'logicism.' For this he needed a definition of 'number' (in the sense of 'positive integer'), which he tried to provide in his famous monograph *The Foundations of Arithmetic* (1884).

> How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a sentence that words have any meaning, our problem becomes this: To define the sense of a sentence in which a number word occurs. (Frege 1884: § 62)

This illustrates Frege's 'linguistic turn,' foreshadowing and inspiring twentieth century analytic philosophy: the question how we come to know numbers is transformed into one about the meaning of sentences in which number words occur. No further intuition or idea is needed or even possible. The quotation also states Frege's 'context principle': that only in the context of a sentence does a word have meaning. We have already seen that it makes no sense to ask for the meaning of 'red' if we do not know whether it occurs as function or as argument. Only in a sentence can we discern the grammatical role of its elements, and thus their meaning. As well, to determine the meaning of a word, one must know whether or not it occurs in an opaque context.

To give a definition of number, Frege used 'Hume's Principle': "When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal" (*Foundations* § 63, Hume, *Treatise of Human Nature* 1.3.1). Plainly, though true and obvious, this is not a principle of *logic*. He therefore tried to deduce it from what he took to be such a principle, the notorious Fifth Principle (in addition to the four of *Begriffsschrift*) which he introduced in his later work, *The Basic Laws of Arithmetic* of 1894. This is the so-called unrestricted comprehension (or abstraction) axiom, to the effect that any concept determines a set that has as its elements the objects that fall under the concept. While he expressed some uneasiness about the principle, he thought it a law of logic that one always has in mind when speaking about the extensions of concepts. Bertrand Russell discovered that a paradox (which bears his name) results from this. The concept '( ) is not a horse' determines the set of all objects not a horse, which includes that set itself. It is thus a set that has itself as an element. Consider now the set S determined by the predicate '( ) is not an element of itself'. If S is an element of itself, then it is not. But if S is *not* an element of itself, then it is, a contradiction from which in Frege's and all 'classical' systems of logic any conclusion whatever follows, rendering the system worthless. A postscript to the second volume of his *Basic Laws* (1903) states:

> Nothing can be more unwelcome to a scientific author than that, after the conclusion of his work, one of the foundations of his building is made to crumble. A letter from Mr. Bertrand Russell placed me in this situation just as the printing of this volume was almost finished. (Frege 1903)

Russell's discovery showed that the axioms of arithmetic (now commonly stated in the form Guiseppe Peano gave them) cannot be formally and consistently derived from Frege's principles (to say nothing of *all* of arithmetic, which cannot be so derived even *given* the axioms (Gödel 1931). But only in recent years has it been shown that these axioms follow from the principles of logic (minus the ill-fated Fifth) together with Hume's Principle. This is now called 'Frege's Theorem.'

7 The Austrian School

Franz Brentano (1838–1917), observed that all 'psychological phenomena' are targeted on some object: when we think, we think of *something*, when we value, we value *something*. These are *intentional objects* whose existence or nonexistence need not be an issue. Brentano shied away from allowing the contents of mental acts to have a form of being, taking this to be an unseemly Platonism. But his students Kasimir Twardowski (1866–1938) and Edmund Husserl (1859–1938) did just that, following Bolzano. Both distinguished *content* from *object*, with the object determined by the content. This is a distinction analogous to Frege's between sense and reference. Although they used figures of speech like the mind *grasping* its objects, they did not draw on psychological theories, and must be absolved of psychologism. Students of Twardowski formed the distinguished school of Polish logicians of the first part of the twentieth century. Of their many achievements we mention only Lesniewsky's (1886–1939) exploration of *mereology* of 1916, a subject that has only recently come to greater prominence. He distinguished the part–whole relation from that of class membership: an element of a class is not a 'part' of it, though a subset is. Importantly, membership is not transitive: if s is an element of t, and t of u, then s is not an element of u, whereas a part of a part is a part of the whole.

Alexius Meinong (1853–1920), another of Brentano's students, inquired into the nature of intentional acts that lack existing objects and are 'beyond being and non-being.' When we think or speak of Hamlet, the content does not refer to a mental image, but to a 'subsisting' object that has lots of properties and satisfies certain identity conditions: the same person killed Polonius and loved Ophelia. Such talk does not lack logical structure. Meinong has more recently been credited with inspiring *free logic*: a logic without existence assumptions, and work in the logic of fiction. For a long time, however, he was known only in caricature through Bertrand Russell's famous article "On Denoting" (1905).

8 Bertrand Russell

In 1905 Russell published "On Denoting," his finest philosophical essay, as he thought. It became a milestone in the development of analytic philosophy. A distinction is here made between proper names and expressions like 'the so and so,' which he titled *definite descriptions*. In English grammar, 'The present king of France is bald' has the subject 'the present King of France' and the predicate 'bald.' But this is misleading. According to Russell, a proper understanding should distinguish three components of its meaning: (1) there is now at least one King in France (2) there is now at most one king in France and (3) every object satisfying (1) and (2) is bald. The sentence is true if all three conditions are satisfied, false if there is no king, if there is more than one king, or if there is a single non-bald king. But if this is what the sentence says, then 'the present king of France' is not part of its proper logical phrasing; a language constructed to strict logical standards will not contain a symbol for it. The misleading 'surface structure' of the sentence disguises its underlying logical structure.

Russell's conclusions are these: (1) Definite descriptions are not names, as Frege had thought; if they were, there would have to be objects to which they refer, leading to Meinong's ontological excesses. (2) Natural language structure and grammar are misleading and must be distinguished from the deeper logical structure. This was a landmark discovery, leading many philosophers to argue that metaphysical and even political convictions often gain their plausibility from deceptive natural language expressions. (3) Expressions like definite descriptions, but not only they, can be defined only in their contexts, by *definitions in use*. 'The present king of France' is not treated as a stand-alone expression and given an 'explicit' definition. Rather, the meaning and function of such expressions is conveyed through the analysis of the sentences in which they occur. (4) It is not necessary, as Meinong had thought, to populate the world with nonexisting, merely *subsisting* objects as the referents of definite descriptions. But there are problems. Some apparent names are disguised descriptions: 'Hamlet' is short for 'the Prince of Denmark'. Unfortunately, then, 'Hamlet loves Ophelia' is just as false as 'Hamlet loves Desdemona', since the prince is fictional. Rather than accept this one might wish to introduce a fictional, subsisting object to answer to the 'Hamlet'.

Despite his discovery of the paradox, Russell held that logicism could be made to work, if the comprehension axiom were restricted. He proposed several solutions, eventually the *theory of types*, fully articulated in the monumental *Principia Mathematica* authored by Russell and A. N. Whitehead (1910–13, three volumes, 1,000 pages), through which Frege's contributions entered the mainstream of logic. The preface states that "in all questions of logical analysis our chief debt is to Frege."

The theory of types stratifies expressions in a hierarchical order so that elements of a set are on a lower level than the set, making it impossible for a set to be a member of itself. A 'ramified' theory of types is introduced to solve as well the so-called semantic paradoxes, notably the liar paradox 'what I now say is false'. Russell and Whitehead were more successful in this than Philetas of Cos (third centuy BC) whose gravestone reads "I am Philetas; the lying argument has killed me and the night – long pondering," and more succinct than Chrysippus, who wrote 28 volumes on it (now lost: Bochenski 1961: 131). But their theory was burdened by the need to recognize a separate definition for truth at each type level and the inability to define a number as the set of all similar (two membered, three membered, etc.) sets. Strictly speaking, every level has different 2s, 3s, 4s, etc., and strictly speaking also different logical principles. They resolve this by using symbols that are 'systematically ambiguous' between types. Further complex adjustments were needed, the axioms of reducibility and choice, which are less than intuitively obvious as they should be for logicism really to succeed. It was also supposed that the vast remainder of mathematics could somehow be reduced to arithmetic, which seems ever more unlikely.

Russell and Whitehead did succeed, however, in deriving a significant portion of mathematics from their principles: a comprehensive theory of relations and order, Cantor's set theory, and a large portion of (finite and transfinite) arithmetic. *Principia* was also meant to be a kind of *Lingua Universalis*, a canonical language pure enough to permit construction of disciplined discourse on the skeleton it provided. Its symbolism was universally accepted, revisions to it addressing problems of readability rather than substance. Some philosophers went farther and proclaimed it the 'ideal language': either translate your claims into *Principia* notation or admit that they are meaningless.

We saw that several distinct areas of study were advanced under the name of logic. There was the view that logic investigates cognitive performance, or else scientific methodology and strategy of discovery, or that it is a branch of rhetoric. Setting aside all these as having contributed little to formal logic as now understood, there were still two distinct types of theory. Until *Principia*, and culminating in that work, the most prominent of them was proof theory, the development of mathematically rigorous *syntactical* procedures for deriving theorems from assumptions. Bolzano, representing the other type of theory, gave a *semantic* definition of logical consequence, which does not dwell on the process of derivation.

The most important development of logic after *Principia* was to bring these two strands together. In propositional logic, for instance, truth tables (introduced by Wittgenstein in 1922) allow a *semantic* test for the validity of formulas and proofs, a continuation of Bolzano's project. It was then proved that the *Principia* version of propositional logic is complete, that is to say that every semantically valid formula can be derived in it and that it is consistent, that is, that only such formulas (and hence no contradiction) can be derived. Later Kurt Gödel proved that first order predicate logic is complete as well, but that higher order logic is not. Since the latter is needed to define arithmetic concepts, this spelled the end of the logicist project.

References

Ashworth, E. J. (1974) *Language and Logic in the Post-medieval Period*. Dordrecht and Boston, MA: Reidel. (Synthese Historical Library, vol. 12.)

Bentham, George (1827) *Outline of a New System of Logic*. London.

Bochenski, I. M. (1961) *A History of Formal Logic*. Notre Dame, IN: University of Notre Dame Press.

Bolzano, Bernard (1837) *Wissenschaftslehre.* Sulzbach: Seidel. (Cited from the English translation by Rolf George (1972) *Theory of Science*. Berkeley and Los Angeles: University of California Press; Oxford: Blackwell.)

Boole, George (1952) *Collected Logical Works*. La Salle, IL: Open Court.

Cranston, Maurice William (1957) *Locke.* New York: Macmillan.

Frege, Gottlob (1879) *Begriffsschrift*. Halle: Nebert. Translated as *Conceptual Notation*, with biography and introduction by Terrell Ward Bynum. Oxford: Clarendon Press, 1972.

Frege, Gottlob (1884) *Grundlagen der Arithmetik*. Translated as *The Foundations of Arithmetic: A Logico-mathematical Enquiry into the Concept of Number* by J. L. Austin. Oxford: Blackwell, 1959.

Frege, Gottlob (1892) Ueber Sinn und Bedeutung. *Zeitschrift fuer Philosophie und Philosophische Kritik*, 100, 25–50. Translated as *On Sense and Reference* in Michael Beaney (ed.), *The Frege Reader*. Oxford: Blackwell, 1997.

Frege, Gottlob (1894, 1903) *Grundgesetze der Arithmetik*. Jena: Pohle. Trans. as *The Basic Laws of Arithmetik* by Montgomery Furth. Berkeley and Los Angeles: University of California Press, 1967.

George, Rolf (1983) Bolzano's consequence, relevance and enthymemes. *Journal of Philosophical Logic*, 12, 299–318.

Gödel, Kurt (1931) On formally undecidable propositions in *Principia Mathematica*. In *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, ed. Jean Van Heijenoort. Cambridge, MA: Harvard University Press, 1967.

Kant, Immanuel (1992) *Lectures on Logic.* Translated and edited by J. Michael Young. Cambridge: Cambridge University Press.

Kneale, William and Martha (1962) *The Development of Logic.* Oxford: Clarendon.

Meinong, Alexius (1902) *Ueber Annahmen*. Reprinted in *Gesamtausgabe*, ed. Rudolf Haller and Rudolf Kindlinger. Graz: Akademische Druck und Verlagsanstalt, 1968–78, vol. IV. Trans. as *On Assumptions*, by James Heanue. Berkeley and Los Angeles: University of California Press, 1983.

Mill, John Stuart (1843) *A System of Logic.* New edition by J. M. Robson in *Collected Works*. Toronto: University of Toronto Press; London: Routledge, 1963–, vols. VII and VIII.

Mill, John Stuart (1865) *An Examination of Sir William Hamilton's Philosophy*. New edition by J. M. Robson in *Collected Works*. Toronto: University of Toronto Press; London: Routledge, 1963–, vol. IX.

Peirce, Charles Sanders (1931) *Collected Papers*, ed. Charles Hartshorne and Paul Weiss. 3rd edn. Cambridge, MA: Harvard University Press, 1974. Vol. II: *Elements of Logic*.

Russell, Bertrand (1905) On denoting. *Mind*. Reprinted in *Logic and Knowledge*, ed. Robert Charles Marsh. London: George Allen & Unwin, 1956.

Whately, R. (1827) *Elements of Logic*. New edition ed. Paola Dessì. Bologna: Editrice CLUEB, 1988.

Zalta, Edward N. (1999) Frege. *The Stanford Encyclopaedia of Philosophy*, http//plato.stanford.edu.