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Introduction

The links between philosophy and mathematics are ancient and complex. The two disciplines are in a sense coeval: for both, the Ancient Greeks were the first to introduce systematicity, rigor, and the centrality of justification to their practice. Indeed, Plato (428–348(7) BCE) had it inscribed on the gates of his Academy that no one should enter who knew no mathematics. There have since been few major philosophers in the Western tradition who have not labored mightily to understand the phenomenon of mathematics.

Yet it might at first seem a surprise that philosophy, which often concerns itself with fundamental issues arising out of everyday life, should focus so on perhaps the most abstract discipline, one whose subject matter and methods seem, as we shall soon see, so removed from ordinary experience. While there is surely no one explanation for this, and certainly no one answer that can be summarized without some injustice, we might point to at least the following.

Our knowledge of mathematical facts is often thought to be justified by pure ratiocination, perceptual observation via any of our five sensory modalities being neither necessary nor even relevant. Because mathematical knowledge seems to be unconstrained by experience, and so to enter the world touched only by the hand of reflection, analysis of it promises to reveal something important about rational thought itself. Since the nature of rationality has always gripped the philosophical imagination, it is perhaps no mystery after all that philosophers regularly direct their attention to mathematics. Mathematics is the purest product of conceptual thought, which is a feature of human life that both pervasively structures it and sets it apart from all else.

Other reasons for philosophical interest in the nature of mathematics will emerge as we proceed. It might be useful to pause, however, and to ask what distinguishes a philosophical interest in mathematics from other kinds of concern. Although the question of what counts as such an interest admits of no briefer or simpler an answer than does that about what philosophy itself is, we can approach it helpfully, if indirectly, by first describing the kinds of study of mathematics we are *not* interested in here.

To begin with, we shall not engage in a *historical* investigation into the development of mathematics. The etiology of mathematical ideas, however interesting, is not something whose study promises to reveal much about the structure of thought: for the most part, the origin and development of mathematical ideas are simply far too determined by extraneous influences. The same holds for a *sociological* inquiry into the role of mathematics and mathematicians in our society, the forces that shape research interests and structure professional activity, and so on: again, these aspects of the doing of mathematics bear the imprint of countless varied factors and so it is difficult, if not impossible, to distill from such inquiries information pertinent to central philosophical concerns.

Since these concerns focus on the nature of rational thought, it might appear especially strange that we are likewise not going to pursue any kind of *psychological* inquiry into mathematical thinking or development. Such research often takes for granted a conception of thought, usually undeveloped, and asks such questions as "What brain, or neural activity, or cognitive architecture makes mathematical thought possible?" or "What kind of environment is needed to facilitate the development of the capacity for such thought?" Again, while of great interest, such studies focus on phenomena that are really extraneous to the nature of mathematical thought itself. Indeed, to repeat, they often proceed without a developed account of what such thoughts are, and they concentrate rather on the neural states that somehow carry thought, or on the environmental or genetic factors that make those states realizable. Philosophers, by contrast, are interested in the nature of those thoughts themselves, in the content carried by the neural vehicles (if that is indeed the right picture). A philosophical study of mind is interested in an analysis of the thoughts that the workings of mind give one access to, but not in an account of the conditions or the mechanisms – environmental, genetic, neurophysiological – to whose influence and operation we owe our access to such thoughts.

We should add, finally, that a philosophical inquiry into mathematics differs from a *mathematical* one. (It is interesting to note that perhaps the only discipline other than philosophy so clearly amenable to such self-analysis, so fitted to take its own methods and problems as a focus of inquiry, is mathematics.) What precisely the application of mathematics to itself might consist in is something that we shall take up later. And although we shall then find this self-application of great interest, its import is not philosophically transparent. Rather, as we shall see, it provides data for a satisfying philosophical analysis, but it cannot substitute for that analysis.

Let us turn now to a more positive characterization of a philosophical approach to mathematics. It will be helpful to focus on instances of actual mathematics, so let us consider a few theorems and their proofs, and then survey the kinds of typically philosophical issues they raise. The first two both involve the distinction between rational and irrational numbers. A *rational* number is one that can be expressed as a fraction; for example, 3/5, −19/12, and 8/1 are all rational numbers. An *irrational* number – π, for instance – is one that cannot be expressed as such a fraction. The rationals and irrationals together make up the *real* numbers. Our first theorem dates from Ancient Greece; it is usually atttributed to a member of the school of Pythagoras, although precisely who first proved it is not known. The English mathematician G. H. Hardy (1877–1947) called it a theorem "of the highest class. [It] is as fresh and significant as when it was discovered – two thousand years have not written a wrinkle" on it.[1]

THEOREM 1.1. *$\sqrt{2}$ is irrational.*

Proof. By *reductio ad absurdum*. Assume that $\sqrt{2} = a/b$, for some integers a, b. By reducing the fraction a/b to lowest terms if necessary, we may assume that a and b have no common factor. Then $2 = (a/b)^2$. That is, $2 = a^2/b^2$. Hence, $2b^2 = a^2$. But then a^2 is even. So, since the square of an odd number is always odd, a is even; that is, $a = 2c$, for some c, and $a^2 = 4c^2$. Substituting, we get that $2b^2 = 4c^2$; that is, $b^2 = 2c^2$. Hence, b is even. But this contradicts our assumption that a and b have no common factor (since if they were both even, they would have a common factor of 2). Therefore, $\sqrt{2} \neq (a/b)$, for any integers a, b; that is, $\sqrt{2}$ is irrational. ■

This demonstrates that not all magnitudes – in particular, not the length of the hypotenuse of a right-angled triangle of unit base and height – can be treated by the theory of numerical proportion upon which the mathematics of Ancient Greece was based. It must have thus constituted something of a revolution.

Our second example is of more recent vintage.[2]

1 Hardy (1967, 92).
2 The earliest published reference known to us is Jarden (1953).

THEOREM 1.2. *There are irrational numbers a, b such that a^b is rational.*

Proof. By argument by cases. Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. By Theorem 1.1, we know that $\sqrt{2}$ is irrational; so let $a = b = \sqrt{2}$. Then a, b are irrational and a^b rational, as required.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is not rational. Let $a = \sqrt{2}^{\sqrt{2}}$ and let $b = \sqrt{2}$. Then $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.
In either case, then, the theorem is shown to be true. ■

Finally, we give an example of a proof that uses *mathematical induction*. Mathematical induction is an important method for proving general statements about the natural numbers: the numbers 0, 1, 2, and so on. To prove a statement of the form "For all natural numbers n, $P(n)$" by mathematical induction, one first proves that $P(0)$ is true (the *base case* of the induction), and one then proves that for every natural number n, if $P(n)$ is true, then so is $P(n+1)$ (the *induction step*). The assumption in the induction step that $P(n)$ is true is called the *inductive hypothesis*. To see why this establishes the desired conclusion, note that according to the induction step with $n = 0$, if $P(0)$ is true then $P(1)$ is true. But by the base case, $P(0)$ *is* true, so it follows that $P(1)$ is also true. Applying the induction step with $n = 1$, we see that if $P(1)$ is true then $P(2)$ is true; since we have just shown that $P(1)$ is true, $P(2)$ is also true. Continuing in this way, we can establish $P(n)$ for every natural number n. In other words, the statement "For all natural numbers n, $P(n)$" is true.

THEOREM 1.3. *For every natural number n*, $0 + 1 + 2 + \ldots + n = n(n+1)/2$.

Proof. By mathematical induction.

Base case: When $n = 0$, both sides of the equation are equal to 0.

Induction step: Suppose that $0 + 1 + 2 + \ldots + n = n(n+1)/2$. Then

$$\begin{aligned} 0 + 1 + 2 + \ldots + (n+1) &= (0 + 1 + 2 + \ldots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$
$$= \frac{(n+1)((n+1)+1)}{2}. \qquad \blacksquare$$

These examples are in a way paradigmatic mathematics. To be sure, mathematics is filled with proofs that are much longer and more complicated, and with theorems that involve concepts far more intricate than those appearing above. But most philosophical questions about mathematics can already be raised with regard to such simple examples. We shall briefly examine a few of these in turn.

To begin with, note that (assuming you had not seen these proofs before) you now know three more truths than you did a few moments ago. How did you acquire this knowledge? One way you did *not* acquire it is through observation. It is true that you used your eyes to read the sentences of the proofs. But this was by no means necessary. You could, after all, have thought up the proofs yourself (as did those who first arrived at them). And anyway, seeing that the ink is arrayed just so on the page is not what shows you that the theorems are true, for the theorems are not about the disposition of ink on paper.

In this connection, philosophers often distinguish between *a priori* and *a posteriori* knowledge. The latter kind of knowledge requires sensory experience for its justification, whereas the former requires none. Mathematical truths, unlike, say, truths about the natural world, are known *a priori*. You now know that $\sqrt{2}$ is irrational, not on the basis of any measurements or observations, but rather on the basis of pure reflection. This is quite unlike your knowledge that this book weighs less than you do, which is *a posteriori* in that it cannot possibly be justified without some kind of recourse to observation.

To be sure, it might be the case that sensory experience is required in order for us to acquire the language by means of which we can understand the thoughts involved in the above theorems and their proofs. But we must distinguish between what is needed in order for one to grasp the content of these theorems and what is relevant to their justification. The present distinction between *a priori* and *a posteriori* knowledge pertains to the latter only. For example, there is a sense in which you would not have known that Theorem 1.1 holds had (say) page 3 been missing from this book (assuming, still, that you learned it here for the first time); sense experience, in particular your sensory interaction with the appropriate page, is relevant to your *acquisition* of the knowledge that this theorem obtains. The physical page and its properties are not, however, relevant to your *justification* for believing that Theorem 1.1 is true. By contrast, the fact that any fraction can be reduced to lowest terms *is* something upon which the justification of Theorem 1.1 depends. One might express this by saying that, should the claim about reducing fractions to lowest terms not hold, then we would not be said to know that Theorem 1.1 is true. But one should not assimilate this situation to that arising from the missing page. The latter leads to an impediment to our acquisition of knowledge, whereas the former presents an obstacle to our justification. To determine whether a claim is known *a priori* or not, we must examine the kind of evidence upon which the justification of the claim rests; in particular, we must determine whether any of it is based on sensory experience. Theorem 1.1, for example, is an instance of *a priori* knowledge because, in spite of the fact that sensory experience is required in order to come to know it, its justification does not depend on any claims known through experience. The same holds for Theorems 1.2 and 1.3.

The above theorems, then, are known *a priori*. Now this feature of mathematical knowledge, that its justification makes no reference to facts known through observation, appears at first glance to clash with *empiricism*, a most influential and long-lived doctrine according to which all our knowledge is ultimately based on our sensory experience. Of course, if “based on” means merely “would not be possible without,” then empiricism cannot be gainsaid any more than other patent truths. So understood, it is not quite a tautology, for there is perhaps nothing absurd in supposing that a being could come to acquire a language and knowledge on the basis of no experience at all, but even the most cursory examination of how it is with humans will show one that it is true. Consequently, if empiricism is to rise above the more than obvious, then “based on” must be interpreted otherwise.

In fact, the doctrine is typically interpreted more strongly to hold that all knowledge is ultimately justified on the basis of sensory experience. And it is with this familiar interpretation of the doctrine that the *a priori* nature of mathematical knowledge is in conflict. The clash is a serious one in so far as empiricism is a plausible doctrine. And it is not hard to see why many have found it such. For much of our knowledge does seem to be rationally based on observations of the world. It might even appear difficult to see how one could have knowledge of the external world except through its rationally impinging upon one in some way. Since our only channels of information are the five sensory modalities, it seems that all knowledge of the world must ultimately be justified by data that are transmitted through them. And yet knowledge of the above three theorems is not.

One way of alleviating the present pressure is to reject the view that these theorems are about the world. Empiricism, after all, appears to be motivated by considering how knowledge of the external world could be justified. Perhaps, then, the entire conflict can be defused by denying that mathematics tells us anything about the natural world. But if it is not about the natural world, then what *is* the subject matter of its claims? One very powerful answer is that mathematics concerns a world that exists quite as independently of us as does the natural world, only one that is not located anywhere in space or time. Hardy eloquently describes this doctrine as follows:

> By physical reality I mean the material world, the world of day and night, earthquakes and eclipses, the world which physical science tries to describe.
>
> I hardly suppose that, up to this point, any reader is likely to find trouble with my language, but now I am near to more difficult ground. For me, and I suppose for most mathematicians, there is another reality, which I will call "mathematical reality." [...] I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply our notes of our observations.[3]

This view of mathematics is often called *platonism*.

Platonism, as just described, in fact involves two distinct ways of characterizing a realm of reality. The first is *ontological* in nature, that is, it proceeds by describing the kind of entities that inhabit the particular realm in question. Mathematical platonists usually insist that mathematical entities are abstract in not being spatiotemporally located and thus in not having any causal powers. If we think about this aspect of platonism in connection with our three theorems, we cannot but acknowledge its plausibility. For instance, it would be quite odd for someone to think that $\sqrt{2}$ is actually a locatable and datable entity. The request for its location or for the time it came into existence admits of no answer that does not confuse a vehicle for referring to that number (for example, ink marks on paper, brain states of a human) with the number itself. Likewise, it seems implausible that the infinitude of natural numbers should remain an open question until physics determines whether the universe is infinite; but if we insist on identifying each of the infinitely many natural numbers with a distinct spatiotemporally located bit of matter, then this is precisely what we shall have to accept.

Platonism provides a second way of characterizing the realm of mathematics, one that we might call *doxastic* in that it describes the relation between, on the one hand, truths about the realm in question and, on the

[3] Hardy (1967, 122–4).

other, what we believe. Along this dimension, platonism insists that mathematics is mind-independent, in the sense that whether a mathematical statement holds is quite independent of what we think. We can imagine certain realms in which the beliefs of observers in effect settle what is true and what is not. But mathematics, according to the platonist, is not like this: the truth or falsity of a mathematical claim is not determined by what anyone believes about its truth value. This, too, is a plausible position with regard to the theorems above. For instance, the square root of 2 is irrational regardless of whether anyone believes or wants it to be; indeed, its irrationality is not contingent on anyone's having beliefs about it at all. This result obtains, Hardy insisted, "not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way."[4]

There are other senses of mind-independence with which this last should not be confused. One might claim that mathematics is mind-independent in that the entities that make up its subject matter are distinct from items of the mind. (Whether this is entailed by the ontological component of platonism depends on whether mental items are themselves spatiotemporally located.) Or one might hold that mathematical entities are mind-independent in that their existence does not presuppose the existence of minds. (The latter conception of mind-independence clearly entails the former. But not conversely: for someone could consistently maintain that just as an artifact, although an object distinct from the artisan, is dependent for its existence on the latter's activities, so mathematical entities, although not themselves mental, would nevertheless not exist were it not for minds.) We shall return to some of these notions later, but for now let us simply observe that they are both to be distinguished from the claim of mind-independence that we are taking to be part of platonism. For even if mathematical entities are identical to mental items, one can still maintain that truths about these entities hold regardless of our beliefs about them: there is no reason why facts about our minds must be in principle accessible to us. We might note, finally, that a generalization of this last observation shows that the two features of platonism we have isolated are distinct: that the truths of mathematics hold regardless of our recognition that they do fails to settle the ontological nature of the entities these truths concern.

We first encountered platonism as a way of viewing mathematics that defuses its threat to empiricism. But does it really succeed in this? One might well wonder whether talk of abstract entities is less a solution to

[4] Hardy (1967, 130).

the empiricist's problem of how *a priori* knowledge is possible than it is a label for the problem. For it remains to be explained how we, creatures located in space and time, can acquire information about entities that are not so located. Because these latter are causally impotent and so cannot affect us in any way, it seems a mystery how we could come to have knowledge about them. To say that thought happens to permit direct access to this independently existing, though causally quarantined, world might seem, again, less a solution than a colorful description of the originally puzzling situation.

In addition, even if one does grant that mathematics is about some nonphysical reality, the fact remains that mathematical knowledge is very *applicable* to physical reality. Mathematics, from the humblest of arithmetical calculations to the most recondite of theories, finds itself useful in the description and prediction of natural phenomena. Physicists are of course particularly aware of this. Albert Einstein (1879–1955) wrote of "an enigma [that] presents itself which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?"[5] As the American physicist Steven Weinberg (1933–) put it, "It is positively spooky how the physicist finds the mathematician has been there before him or her."[6] Johannes Kepler (1571–1630), the German astronomer and mathematician, likewise impressed by this fact, suggested that "God himself was too kind to remain idle, and began to play the game of signatures, signing his likeness into the world; therefore I chance to think that all nature and the graceful sky are symbolized in the art of geometry." If the applicability of mathematics calls for some explanation, then the mystery is only deepened by taking mathematics to be about a realm disjoint from the natural world.

Faced with these initial hurdles for platonism, one might be tempted to rethink one's strategy for safeguarding empiricism from the *a priori* nature of mathematical knowledge. And there is indeed another measure to hand, namely the proposal that, contrary to all appearances, mathematical truths are in fact known *a posteriori*. Although some philosophers have attempted to elaborate such a view, it is hard to see how they can overcome the fact already emphasized: that mathematical arguments do not seem to be justified by any empirical observations. In addition, the

[5] From a lecture delivered to the Prussian Academy of Sciences in Berlin on January 27, 1921: see Einstein (1983, 28).

[6] Weinberg (1986).

very nature of justification in mathematics, namely proof, appears quite different from that usually operative in arguments to conclusions about the natural world. The latter inferences are often inductive or statistical in character, and so not truth-preserving: that is, the truth of their premises does not guarantee the truth of their conclusions, but may make them merely probable to some degree or other. By contrast, the inferences found in mathematical proofs are all *valid*: the truth of such an inference's premises necessitates the truth of its conclusion. It seems, then, that mathematical truths not only fail to be known on the basis of empirical evidence, but they also fail to be justified by inferences of the kind typically used to reason about the natural world. This proposal really does appear a counsel of despair.

This last point draws our attention to yet another feature of mathematics that has puzzled philosophers: the validity of the inferences that figure in its deductions. Consider the form of inference at work in Theorem 1.1. We can analyze it as having the following structure:[7]

(1) $A \to \neg A$.

Therefore,

(2) $\neg A$.

This argument, known as *reductio ad absurdum*, has the distinctive property that if (1) is true, then (2) must be true as well.[8] The claim is not merely that (2)'s truth is made very likely by the truth of (1) but, rather, that it is made necessary. Although the proof of Theorem 1.2 is supported by a different logical skeleton, it shares this property. Its structure is as follows:

(3) $A \vee \neg A$,

(4) $A \to B$,

[7] In this book, we use the following logical notation: "$\neg P$" means *not P*, "$P \wedge Q$" means *P and Q*, "$P \vee Q$" means *P or Q*, "$P \to Q$" means *if P then Q*, "$P \leftrightarrow Q$" means *P if and only if Q*, "$\forall x P$" means *for all x, P*, and "$\exists x P$" means *there exists at least one x such that P.*

[8] Hardy memorably said of this form of inference that it "is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*." (Hardy, 1967, 94).

(5) $\neg A \rightarrow B$.

Therefore,

(6) B.

The unique force of the "therefore" here is just this: that should (3)–(5) be true, then necessarily (6) is as well. The Austrian philosopher Ludwig Wittgenstein (1889–1951) called this the "hardness of the logical *must*,"[9] and many have found its source quite mysterious. For instance, the Scottish philosopher David Hume (1711–76) argued that such a necessary connection between states of affairs is illusory, for nothing like it is given to us in experience; at best, we observe that one kind of state of affairs regularly follows another, but the *necessity* of this link is nowhere to be seen.[10] This necessity is one source of the aesthetic fascination that mathematics exerts. The English philosopher Bertrand Russell (1872–1970) once told a story that illustrates this:

> My friend G. H. Hardy, who was Professor of pure mathematics, enjoyed this pleasure in a very high degree. He told me once that if he could find a proof that I was going to die in five minutes he would of course be sorry to lose me, but this sorrow would be quite outweighed by pleasure in the proof. I entirely sympathized with him and was not at all offended.[11]

The bewitchingly mysterious necessity of proof highlights yet another aspect of mathematics that is difficult to accommodate within an empiricist perspective.

There is a final feature of mathematics, already obliquely mentioned, that deserves comment here. This is that the knowledge justified by mathematical proof is often *infinitary* in nature: the content of what is known involves reference to an infinite range of objects. Consider, for example, Theorem 1.1, whose infinitary content might not at first be obvious, since it seems to be about just one entity, the square root of 2. The fact is that the property being attributed to this entity, that of irrationality, when rendered fully explicit, makes reference to an infinite collection. For to say that $\sqrt{2}$ is irrational is just to say that

$$\neg\exists x \exists y(x \text{ is an integer} \wedge y \text{ is an integer} \wedge \sqrt{2} = x/y).$$

[9] Wittgenstein (1978).
[10] See, for instance, Hume (1748).
[11] Russell (1956, 14).

And this is logically equivalent to

$$\forall x \forall y((x \text{ is an integer} \wedge y \text{ is an integer}) \rightarrow \sqrt{2} \neq x/y).$$

As the universal quantifiers ranging over the totality of integers reveal, the claim that $\sqrt{2}$ is irrational is an infinitary one. Now this feature of mathematics can appear puzzling. We are, after all, finite creatures: our powers, such as our memory and computational speed, are finite, as are the durations for which we can exercise these powers, namely our lives. Consequently, we have the capacity to survey no more than a finite amount of evidence. How, then, do we manage to arrive at infinitary knowledge? One might even begin to find it mysterious that we finite beings are able so much as to *comprehend* statements about an infinite range of objects. As the French writer Voltaire (1694–1778) said, the infinite "astonishes our dimension of brains, which is only about six inches long, five broad, and six in depth, in the largest heads."[12]

Mathematics, in short, seems to be a discipline through which we acquire knowledge about infinitely many entities with which we can in no way causally interact, by means of finite inferences which make no use of empirical premises and which yield their conclusions with the force of necessity. We began by noting that it might seem surprising that mathematics should be the object of so much philosophical attention. But perhaps what should surprise is any display of intellectual equanimity before the phenomenon of mathematics.

Philosophers have not been alone in trying to understand mathematics, however distinctive their own perplexities and approaches might be. Mathematicians, spurred especially by dissatisfaction with their understanding of analysis, the mathematics that underpins the calculus, have in their own way sought to arrive at a deeper understanding of the foundations of their discipline. These attempts were especially sustained in the nineteenth century, although the worries dated from before. The Irish philosopher Bishop Berkeley (1685–1753), complaining in his *The Analyst, or a Discourse Addressed to an Infidel Mathematician* (1734) about Newton's formulation of the calculus, quipped that: "He who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in divinity."[13] Although many struggled to clarify these matters, the mathematician N. H. Abel (1802–29) could still complain in 1826, almost a century later, of "the surprising obscurity one finds undoubtedly in analysis today. It lacks all plan and

[12] Quoted in Moritz (1958, 336).
[13] Reprinted in Ewald (1996, vol. 1, 62–92; 65, para. 7).

unity [...] the worst is that it has not at all been treated with rigor. There are only a very few theorems in advanced analysis which are proved with complete rigor."[14] Indeed, at that time mathematicians were still lacking definitions of basic analytic notions such as limit, continuity, the distinction between pointwise and uniform convergence, and the derivative.

Yet by the end of the nineteenth century, conceptual clarity regarding the foundations of analysis had been largely achieved. The rigor whose absence was lamented by Abel was now in place, and it made possible not only a deeper understanding of previous results (and errors), but also suggested new, formerly unthinkable avenues of exploration. This enhanced focus developed as mathematics, the calculus especially, underwent a systematic analysis in terms of the familiar natural numbers and the usual operations defined on them. Of particular note in this connection is the accomplishment, due primarily to the German mathematician Richard Dedekind (1831–1916), of defining the integers, rationals and reals, taking only the system of natural numbers for granted.

It suffices now to say that for some this process of analysis looked to be at an end, on the grounds that there was nothing more basic than the natural numbers to which they in turn could be reduced. In this spirit, the German mathematician Leopold Kronecker (1823–91) famously announced that "God made the integers, all the rest is the work of man."[15] This attitude does not entail that no further progress can be made in understanding the basis of mathematics, but only that we should not look to mathematics itself for deeper illumination; in the natural numbers, we have hit mathematical rock-bottom. Should still greater insight be achievable through a philosophical analysis of natural number, someone with this attitude might urge, we must not expect it also to provide anything that could pass for a mathematical account.

In the next chapter, we shall consider how in fact some sought to achieve both goals simultaneously.

[14] Abel (1902, 23), quoted in Sieg (1984).

[15] Quoted in Bell (1937, 477).