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Introduction: Frege's Life and Work

Biography

Friedrich Ludwig Gottlob Frege was the founder of modern mathematical logic, which he created in his first book, *Conceptual Notation, a Formula Language of Pure Thought Modelled upon the Formula Language of Arithmetic* (*Begriffsschrift, eine der arithmetischen nachgebildete Formalsprache des reinen Denkens* (1879), translated in Frege 1972). This describes a system of symbolic logic which goes far beyond the two thousand year old Aristotelian logic on which, hitherto, there had been little decisive advance. Frege was also one of the main formative influences, together with Bertrand Russell, Ludwig Wittgenstein and G. E. Moore, on the analytical school of philosophy which now dominates the English-speaking philosophical world. Apart from his definitive contribution to logic, his writings on the philosophy of mathematics, philosophical logic and the theory of meaning are such that no philosopher working in any of these areas today could hope to make a contribution without a thorough familiarity with Frege's philosophy. Yet in his lifetime the significance of Frege's work was little acknowledged. Even his work on logic met with general incomprehension and his work in philosophy was mostly unread and unappreciated. He was, however, studied by Edmund Husserl, Bertrand Russell, Ludwig Wittgenstein and Rudolf Carnap and via these great figures he has eventually achieved general recognition.

Frege's life was not a personally fulfilled one (for more detailed accounts of the following see Bynum's introduction to Frege 1972 and Beaney's introduction to Frege 1997). His wife died twenty years before his own death in 1925 (he was survived by an adopted son, Alfred) and, ironically, his life's work in the philosophy of mathematics, to which he regarded all the rest of his efforts as subordinate, that is, his attempted demonstration that arithmetic was a branch of logic (the 'logicist thesis' as it is now called), was dealt a fatal blow by Bertrand Russell, one of his greatest admirers, who showed that it entailed the inconsistency that now bears his name ('Russell's Paradox'). Nevertheless, Frege perhaps gained some comfort from the respect accorded to him by Russell and by Wittgenstein, who met Frege several times and revered him above all other philosophers. In retrospect, indeed, many would perhaps say that in philosophy generally, as distinct from the narrower branches of logic and the philosophy of mathematics, Frege's greatest contribution was the advance in the philosophy of logic and language which made Wittgenstein's work possible.

Little is known of Frege's personality and life outside philosophy. Apparently his politics and social views, as recorded in his diaries, reveal him to have been, in his later years, extremely right-wing, strongly opposed to democracy and to civil rights for Catholics and Jews. Frege's greatest commentator, Michael Dummett, expresses great shock and disappointment (1973: xii) that someone he had revered as an absolutely rational man could have been imbued with such prejudices. But a more generous view is the one expressed by another great Frege scholar, Peter Geach. Geach writes that while Frege was indeed imbued with typical German conservative prejudices, 'to borrow an epigram from Quine, it doesn't matter what you believe so long as you're not sincere. Nobody can really imagine Frege as an active politico devoted to some course like Hitler's' (1976c: 437).

We have, however, a presentation of the more attractive side of Frege in an account Wittgenstein gives of his encounters with him:

> I was shown into Frege's study. Frege was a small, neat man with a pointed beard who bounced around the room as he talked. He absolutely wiped the floor with me, and I felt very depressed; but at the end he said 'You must come again,' so I cheered up. I had several discussions with him after that. Frege would never talk about anything but logic and mathematics, if I started on some other subject, he would say something polite and then plunge back into logic and mathematics. He once showed me an obituary on a colleague, who,

> it was said, never used a word without knowing what it meant; he expressed astonishment that a man should be praised for this! The last time I saw Frege, as we were waiting at the station for my train, I said to him 'Don't you ever find any difficulty in your theory that numbers are objects?' He replied: 'Sometimes I *seem* to see a difficulty but then again I *don't* see it.' (Included in Anscombe and Geach 1961)

Rudolf Carnap, who attended Frege's lectures in 1914, also presents a vivid image:

> Frege looked old beyond his years. He was of small stature, rather shy, extremely introverted. He seldom looked at his audience. Ordinarily we saw only his back, while he drew the strange diagrams of his symbolism on the blackboard and explained them. Never did a student ask a question or make a remark, whether during the lecture or afterwards. The possibility of a discussion seemed to be out of the question. (Carnap 1963: 5)

Frege was born in 1848 in Wismar on the German Baltic coast. He attended the Gymnasium in Wismar for five years (1864–9), passed his Abitur in the spring of 1869 and then entered Jena University.

There Frege spent two years studying chemistry, mathematics and philosophy. He then transferred to the University of Göttingen (perhaps influenced by one of his mathematics professors, Ernst Abbe), where he studied philosophy, physics and mathematics.

In 1873 Frege presented his doctoral dissertation, *On a Geometrical Representation of Imaginary Figures in a Plane* (in Frege 1984: 1–55), which extended the work of Gauss on complex numbers, and was granted the degree of Doctor of Philosophy in Göttingen in December of that year.

Frege then applied for the position of Privatdozent (unsalaried lecturer), at the University of Jena. Among the documents he supplied in support of his application was his Habilitationsschrift (postdoctoral dissertation required for appointment to a university teaching post), 'Methods of Calculation Based upon an Amplification of the Concept of Magnitude' (in Frege 1984: 56–92). In this piece there first emerges Frege's interest in the concept of a *function* which, as we shall see, was to play an absolutely central role throughout his philosophy.

Frege's work was judged acceptable by the Jena mathematics faculty, and in a prescient report Ernst Abbe speculated that it contained the seeds of a viewpoint which would achieve a durable advance in mathematical analysis. Frege was therefore allowed to proceed to an oral examination, which he passed, though he was judged to be neither quick-witted nor fluent. After a public disputation and trial lecture in May 1874 he was appointed Privatdozent at Jena, where he remained for the rest of his career.

Initially Frege had a heavy teaching load and he only published four short articles (see Frege 1984: 93–100), three of them reviews and one an article on geometry, before 1879, when *Conceptual Notation* was published. Nevertheless, these were probably the happiest years of his life. He was young, ambitious, with a plan of his life's work (as we see from the Preface to *Conceptual Notation*) already formed. He was, moreover, well thought of by the faculty and by the best mathematics students at Jena. The description of his 'student friendly' lecturing style quoted from Carnap earlier fits with Abbe's evaluation of Frege for the university officials in 1879. Abbe reported that Frege's courses were little suited to please the mediocre student 'for whom a lecture is just an exercise for the ears'. But 'Dr Frege, by virtue of the great clarity and precision of his expression and by virtue of the thoughtfulness of his lectures is particularly fit to introduce aspiring listeners to the difficult material of mathematical studies – I myself have repeatedly had the opportunity to hear lectures by him which appeared to me to be absolutely perfect on every fundamental point' (quoted in Frege 1972: 8).

Absolute perfection on every *fundamental* point was indeed the aim – and the achievement – of Frege's *Conceptual Notation*, which he conceived as the necessary starting point of his logicist programme. It appeared in 1879 and partly as a result Frege was promoted to the salaried post of special (*ausserordentlicher*) Professor. The promotion was granted on the strength of a recommendation by Frege's mentor Ernst Abbe, who wrote with appreciation of *Conceptual Notation*. His remarks were again prescient. He thought that mathematics 'will be affected, perhaps very considerably, but immediately only very little, by the inclination of the author and the content of the book'. He continued by noting that some mathematicians 'will find little that is appealing in so subtle investigations into the formal interrelationships of knowledge', and 'scarcely anyone will be able, offhand, to take a position on the very original cluster of ideas in this book . . . it will probably be understood and appreciated by only a few' (quotations from Frege 1972: 16).

Abbe's pessimism about the immediate reception of Frege's work was wholly justified. It received at least six reviews, but none showed an appreciation of the book's significance, even though some of the reviewers were eminent logicians. The reviews by Schröder in Germany and Venn in England must have been particularly bitter disappointments. Frege's work was judged inferior to the Boolean logic of his leading contemporaries and his 'conceptual notation' dismissed as 'cumbrous and inconvenient' (by Venn) and 'a monstrous waste of space' which 'indulges in the Japanese custom of writing vertically' (by Schröder).

It was an unfortunate outcome but neither without precedent, nor, in retrospect, surprising. The extent of Frege's achievement was something that could not possibly have been expected by a reviewer asked to give an initial assessment of his work. One is reminded of the similar reception of David Hume's *Treatise of Human Nature* which, likewise, as Hume famously put it, 'fell dead-born from the press'. And, as also in the case of Hume, the poor reception of Frege's work was partly his own fault – arising from the 'manner rather than the matter' of presentation, to use Hume's words – and something that could have been anticipated. Frege did not explain clearly and thoroughly the purpose of *Conceptual Notation* and did not justify and illustrate the advantages of his bizarre-looking two-dimensional notation and its superiority to those available at the time. One can thus sympathize with the first reviewers. As a recent commentator has put it: 'The odds that Frege's work was the production of a genius rather than a crackpot may have seemed long indeed to his colleagues and contemporaries' (Boolos 1998: 144).

As a result of the poor reception of *Conceptual Notation*, Frege postponed his plan, announced in its preface, to proceed immediately to the analysis of the concept of number. Instead he attempted to answer his critics. He wrote two papers comparing his logical symbolism with that of Boole. The first, 'Boole's Logic Calculus and the Concept-Script' (now published in Frege 1979: 9–46) was rejected by three journals. The second, a much shorter version of the first, 'Boole's Logical Formula Language and my Conceptual Notation' (now in Frege 1979: 47–52) was also rejected. Finally Frege managed to get published a more general justification of his conceptual notation, 'On the Scientific Justification of a Conceptual Notation' (now in Frege 1972: 83–9), and was able to deliver a lecture, also subsequently published, at a meeting of the Jenaischüe Gesellschaft für Medicin und Naturwissenschaft, in which he compared his symbolism with Boole's ('On the Aim of the Conceptual Notation', now in Frege 1972: 90–100).

The disappointing reviews of *Conceptual Notation* thus sidetracked Frege into a frustrating episode of self-justification. But they also had the effect of making him more aware of how he must present his work if it was to be appreciated. Instead of proceeding straight from *Conceptual Notation* to a formal demonstration, in his symbolic notation, of the derivability of arithmetic from logic, as anticipated in the Preface to *Conceptual Notation*, Frege decided to produce an informal sketch of his derivation in ordinary German, set out against the background of a critique of traditional (including Kantian and empiricist) views of number. The result was his masterpiece, *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number* (*Die Grundlagen der Arithmetik: eine logische mathematische Untersuchung über den Begriff der Zahl*) published in 1884 (Frege 1968).

Once again, as in the case of *Conceptual Notation*, Frege viewed this only as a preliminary stage in his logicist project. He thought that he had made the 'analytic character of arithmetical propositions' (i.e. their derivability from logical laws by definition) 'probable', but to *prove* his thesis he needed to produce 'a chain of deductions with no link missing' using principles of inference all of which could be recognized as purely logical (Frege 1968: 102).

Frege could have hoped that after *Foundations* his achievement of this project would have been eagerly awaited by scholars. For *Foundations* is indeed, as Frege intended, a brilliantly written exposition of his views, both negative and positive. In fact, it received only three reviews, all of them hostile (one, by Cantor, criticizing Frege on the basis of the misunderstanding that he took numbers to be sets of physical objects), and remained largely unread and unnoted for nearly twenty years. A partial explanation of this situation is perhaps the poor reception of *Conceptual Notation*, which could not have added to Frege's reputation or predisposed mathematicians and philosophers to think his subsequent work worthy of the effort needed to understand it. But whatever the case, the result was that Frege had no choice but to persevere with his logicist project unacknowledged and unsupported by any encouragement from his peers.

The next stage in this project appeared as volume 1 of *The Basic Laws of Arithmetic* (*Grundgesetze der Arithmetik*) in 1893 (see Frege 1962, translated in part in Frege 1964). However, in the intervening nine years Frege's views on the underlying philosophy of language and logic of *Foundations* developed rapidly, necessitating a complete rewriting of a large preliminary manuscript for *Basic Laws*. It was in this period that he published, in the early 1890s, his three best known papers 'Function and Concept' (*Funktion und Begriff*), 'On Sense and Reference' ('Über Sinn und Bedeutung') and 'On Concept and Object' ('Über Begriff und Gegenstand') (all in Frege 1969). All three of these are now regarded as classic works in the philosophy of language, and the second, in particular, must be read by anyone who wishes to understand twentieth-century analytic philosophy at all, but their importance for Frege was that they set out the changes in his views from the time of *Foundations* and prepared their readers for *Basic Laws*.

In this period, also, notice began to be taken notice of Frege's works when the Italian logician Peano cited them in print and Husserl began to correspond with Frege.

With volume 1 of *Basic Laws* written, Frege should now have been able to look forward to its publication and the recognition his work had for so long gone without. However, so poorly had his previous work been received that no publisher would print the lengthy manuscript as a whole. Frege eventually got an agreement from Hermann Pohle of Jena, who had published 'Function and Concept', to publish it in two volumes, with the publication of the second volume being conditional on the success of the first. In this way volume 1 was eventually printed in 1893.

Frege evidently anticipated that his work was likely, once more, to fail to gain the recognition it deserved. He acknowledged that:

> An expression cropping up here or there, as one leafs through these pages, may easily appear strange and create prejudice . . . Even the first impression must frighten people off: unfamiliar signs, pages of nothing but alien looking formulas . . . I must relinquish as readers all those mathematicians who, if they bump into logical expressions such as 'concept', 'relation', 'judgement', think: *metaphysica sunt, non leguntur*, and likewise those philosophers who at the sight of a formula cry: *mathematica sunt, non leguntur*; and the number of such persons is surely not small. Perhaps the number of mathematicians who trouble themselves over the foundations of their science is not great, and even those frequently seem to be in a great hurry until they have got the fundamental principles behind them. And I scarcely dare hope that my reasons for painstaking rigour and its inevitable lengthiness will persuade many of them. (Frege 1964: xi–xii)

For this reason Frege made great efforts to make his work more accessible to his readers. He gave hints in the Preface as to how to read the book to achieve a speedy understanding and in the text he prefaced his proofs with rough outlines to bring out their significance. He also attempted to provoke other scholars to respond to his work by attacking rival theories.

It was all to no avail. Volume 1 of *Basic Laws* received just two reviews, both unfavourable, one of only three sentences, and was otherwise ignored. As a result the publisher refused to publish the remainder of Frege's work and volume 2 eventually had to be published a decade later by Frege at his own expense.

Nevertheless, publication of volume 1 at least led to an improvement in Frege's material circumstances, with his promotion in 1896 to the rank of Honorary Ordinary Professor. This was unsalaried but without administrative duties. Frege was able to accept this post because he was offered a stipend from the Carl Zeiss Stiftung, founded and sustained by his mentor Ernst Abbe. Consequently Frege now had more time for his research, and in the decade preceding the publication of volume 2 of *Basic Laws* engaged in correspondence with a variety of scholars, and published a number of articles and reviews of other authors as well as carrying forward his work on the *Basic Laws*.

One of the scholars Frege corresponded with in this period was Peano, who had written the longer of the two reviews of volume 1 of *Basic Laws* The review started an exchange of views and led Peano to make modifications in his logical symbolism (Frege's correspondence with Peano is published in Frege 1980: 108–29; his two pieces explaining the superiority of his logical notation to that of Peano are published in Frege 1980: 112–18). Another fateful result of Frege's coming to the notice of Peano was that Russell, who adopted Peano's notation, learned of his work. As Russell himself tells the story (Russell 1959: 65):

> I did not read [the *Begriffsschrift*] . . . until I had independently worked out a great deal of what it contained . . . I read it in 1901. . . . What first attracted me to Frege was a review of a later book of his by Peano [Peano's review of volume 1 of the *Grundgesetze*] accusing him of unnecessary subtlety. As Peano was the most subtle logician I had at that time come across, I felt that Frege must be remarkable.

Apart from Peano, another scholar on whom Frege had some influence during this period prior to the publication of the second volume of *Basic Laws* was Husserl, the founder of the continental phenomenological school. Husserl began as a disciple of Brentano and an advocate of psychologism (the attempt to base logic and arithmetic on psychology). In 1891 he published the first volume of his *Philosophy of Arithmetic*. This contained criticisms of Frege and Frege responded in 1894 with a scathing review. Husserl was converted from psychologism and became henceforth its strong opponent, developing the notion of the *noema* of an act of thought, which corresponds to, but is intended to generalize, Frege's notion of sense.

Thus, although his own work was still neglected, Frege could at least take comfort from the fact that he was now known and respected by some of the most eminent scholars of the day, and look forward to a better reception for volume 2 of *Basic Laws*. Despite the neglect of his work, he himself never doubted its achievement. In the final paragraph to the Preface of volume 1 of *Basic Laws* he raises the question of the possibility of someone deriving a contradiction in his system, but dismisses it with total confidence:

> It is *prima facie* improbable that such a structure could be erected on a base that was uncertain or defective. . . . As a refutation in this I can only recognize someone's actually demonstrating either that a better, more durable edifice can be erected upon other fundamental convictions, or else that my principles lead to manifestly false conclusions. But no one will be able to do that. (1964: xxvi)

Disaster struck in the form of a modestly expressed letter from Russell which arrived in June 1902, as the second volume of *Basic Laws* was in press. Russell's letter pointed out that the contradiction now known as 'Russell's Paradox' was derivable in Frege's logical system. After expressing his admiration for Frege's work and his substantial agreement, Russell writes:

> I have encountered a difficulty only on one point. You assert (p.17) that a function could also constitute the indefinite element. This is what I used to believe, but this view now seems to me dubious because of the following contradiction: let *w* be the predicate of being a predicate which cannot be predicated of itself. Can *w* be predicated of itself? From either answer follows its contradictory. We must therefore conclude that *w* is not a predicate. Likewise, there is no class (as a whole) of those classes which, as wholes, are not members of themselves. From this I conclude that under certain circumstances a definable set does not form a whole. (Frege 1969: 130–1)

Frege recognized at once the seriousness of the difficulty Russell had explained and identified Basic Law (V) as its origin. He wrote back to Russell:

> Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck because it has rocked the ground on which I meant to build arithmetic. It seems accordingly that the transformation of the generality of an equality [*Gleichheit*] into an equality of value ranges is not always permissible, that my law (V) is false, and that my explanations do not suffice to secure a reference [*Bedeutung*] for my combination of signs in all cases. I must give some further thought to the matter. It is all the more serious as the collapse of my law (V) seems to undermine not only the foundations of my arithmetic but the only possible foundations of arithmetic as such. (Frege 1969: 132–3)

Frege attempted to develop a response to the paradox and published an amendment in an appendix to volume 2 of *Basic Laws.* However, the amended system can also be proved to be inconsistent (see Quine 1955; Geach 1956) and although it is unclear when Frege finally accepted that his work had been fatally undermined, the third volume of *Basic Laws* was never published and at the end of his life Frege admitted that his logicist programme had been a failure, and attempted in his last years to found arithmetic on geometry.

After the discovery of the paradox Frege was now to suffer personal tragedy. His wife died, leaving him to bring up his adopted son, Alfred, on his own. He published little following this, apart from several articles on the foundations of geometry which arose from his correspondence with Hilbert before the disclosure of Russell's Paradox, and three articles against 'formalist' arithmetic in response to an attack by his Jena colleague Johannes Thomae. However, Frege did engage in extensive correspondence during this period and continued his lectures at Jena. And this was the time that he met Wittgenstein, who wrote to him after reading an account of his views in Russell's *Principles of Mathematics*. The correspondence led to a meeting and as well as discussing his own views with Wittgenstein Frege also made the suggestion that Wittgenstein should go to Cambridge to study with Russell.

It was also during this period that Rudolf Carnap attended Frege's lectures. Like Wittgenstein, Carnap greatly admired Frege's work and developed and disseminated his ideas when he subsequently became influential.

Frege retired from lecturing in 1918 and moved from Jena to Bad Kleinen, near his home town of Wismar. He did not cease working and appears to have gained renewed vigour in this later period. At any rate he wrote a series of papers, of which the first, 'Thoughts' ('Der Gedanke'), has had more influence and attracted more discussion than any of Frege's papers apart from 'On Sense and Reference'. During this time, too, Frege came to believe that arithmetic must have a geometrical foundation. In a piece entitled 'Numbers and Arithmetic' written in the last year of his life he wrote:

> The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis – a geometrical one in fact – so that mathematics in its entirety is really geometry. Only on this view does mathematics present itself as completely homogeneous in nature. Counting, which arose psychologically out of the demands of practical life, has led the learned astray. (Frege 1979: 275–7)

Thus Frege at last abandoned the view he had held ever since his first publication, that arithmetic, unlike geometry, was a source of *a priori* knowledge requiring no foundation in intuition.

However, Frege did not have the time left to pursue his new ideas and he died in 1925, aged seventy-seven, before he was able to know of the widespread influence his work was to have.

He bequeathed his unpublished writings to his son Alfred with the following note attached (now printed in Frege 1979: ix):

> Dear Alfred,
> Do not despise the pieces I have written. Even if all is not gold, there is gold in them. I believe there are things which will one day be priced much more highly than they are now. Take care that nothing gets lost.
> Your loving father.
> It is a large part of myself that I bequeath to you herewith.

Alfred handed over Frege's papers to Heinrich Scholz of the University of Münster in 1935. Unfortunately the originals were destroyed by Allied bombing during the war. However, copies had been made of most of the important pieces and eventually, after a long delay, due to Scholz's own illness and death, they were published in German in 1969 and in English in 1979. Meanwhile Frege's correspondence was edited and published in German in 1976 and in English (in an abridged edition) in 1980.

The Origin and Development of Frege's Philosophy

It is clear that from the start of his career Frege was interested in seeking a foundation for arithmetic. How important he took it to be for a mathematician to be clear about the fundamentals of his subject is made very obvious in a harsh review, his first publication after his appointment as Privatdozent at Jena, of a book on *The Elements of Arithmetic* by one H. Seager (in Frege 1984). Frege writes:

> After some particularly unfortunate explanations of the calculating operations and their symbols, some propositions are presented in the second and third chapters under the title of 'the fundamental theorems and most essential transformation formulas'. These propositions, which actually form the foundation of the whole of arithmetic, are lumped together without proof; while, later, theorems of a much more limited importance are distinguished with particular names and proved in detail. . . . The amplification of concepts which is so highly important for arithmetic, and is often the source of great confusion for the student, leaves much to be desired. . . . The result of all these deficiencies will be that the student will merely memorize the laws of arithmetic and become accustomed to being satisfied with words he does not understand.

Whether it was reading Seager's book that stimulated in Frege the ambition to set arithmetic on pure logical foundations we do not know. But we will understand this project better if we place it in the philosophical and mathematical context of his time. In particular, it will be illuminating to look briefly at the links between Frege's project and the Kantian doctrine of the synthetic *a priori* and the associated notion of pure intuition; the development of non-Euclidean geometry; and the arithmetization of analysis.

For Kant mathematics was an epistemological puzzle, combining two apparently irreconcilable features: necessity and substantiality. Mathematical propositions seem to state truths that could not be otherwise. But at the same time they appear to represent genuine extensions of our knowledge. In this respect, Kant thought, they were like the maxim of universal causation, that every event has a cause, whose problematic status Kant's predecessor Hume, the great British empiricist, had brought to his attention. Hume had operated with a dichotomy, between relations of ideas and matters of fact, which did not allow any place for a proposition of this character. Thus he claimed that the causal maxim was, in fact, a merely contingently true statement of a matter of fact and that our ascribing to it the character of a necessary truth was a mistake whose psychological origin he took it to be one of his principal achievements to have explained. Kant would not accept this, however, but neither was he willing to accept that the causal maxim merely expressed a Humean relation of ideas, and so, like the proposition that every *effect* has a cause, was trivially true in virtue of its meaning. Both in this case and the case of mathematics, Kant thought, what we had to acknowledge was the existence of propositions which fell on neither side of Hume's dichotomy.

Kant discusses this problem in *The Critique of Pure Reason* (1781) within the framework of a pair of distinctions: (i) between *a priori* and *a posteriori* knowledge and (ii) between *analytic* and *synthetic* judgements. He explains the first term of the first distinction as follows:

> We shall understand by *a priori* knowledge, not knowledge independent of this or that experience, but knowledge absolutely independent of all experience. (Kant 1929 A2/B3: 43)

A posteriori knowledge, then, is knowledge that does require experience.

Kant's second distinction he explains as follows:

> Either the predicate B belongs to the subject A, as something which is (covertly) contained in this concept A, or B lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgement analytic, in the other synthetic. (Kant 1929 A6/B10: 48)

In an analytic judgement, Kant says, in thinking the subject term one thinks the predicate term, so no new knowledge can be expressed in an analytic judgement. He illustrates this distinction with the following example:

> Analytic judgement: All Bodies are Extended
> Synthetic judgement: All Bodies are Heavy

Thus for Kant, there are four possible categories of judgement. The synthetic *a posteriori*, the synthetic *a priori*, the analytic *a posteriori* and the analytic *a priori*. The first category, illustrated by the judgement 'All Bodies are Heavy' is unproblematic, as is the fourth, illustrated by the judgement 'All Bodies are Extended', the third category is unproblematically empty. For Kant it is the second category, the synthetic *a priori*, which is of interest. It is in this category that he places the causal maxim and mathematical propositions, as extending our knowledge – since the concept of the predicate is not thought in thinking the concept of the subject – and at the same time as necessary and universally true and, therefore, knowable independently of the contingencies of particular features of our experience.

Kant's argument for the synthetic *a priori* character of mathematics is clearly expressed in his *Prolegomena* (1783) (Kant 1959).

First he argues:

> properly mathematical propositions are always judgements *a priori*, and not empirical, because they carry with them necessity, which cannot be taken from experience. (1959: 18–19)

Next he argues, illustrating his point with his favourite mathematical proposition, that 7 + 5 = 12 is synthetic because twelve can never be found in the analysis of the sum of seven and five:

> The concept of twelve is in no way already thought merely by thinking this unification of seven and five, and though I analyse my concept of such a possible sum as long as I please, I shall never find the twelve in it. We have to go outside these concepts and with the help of the intuition which corresponds to one of them, our five fingers for instance, (or as Segner does in his Arithmetic) five points, add to the concept of seven, unit by unit, the five given in intuition. Thus we really amplify our concept by this proposition 7 + 5 = 12, and add to the first concept a new one which was not thought in it. That is to say, arithmetical propositions are always synthetic, of which we shall be the more clearly aware if we take rather larger numbers. For it is then obvious that however we might turn and twist our concept, we could never find the sum by means of mere analysis of our concepts without seeking the aid of intuition. (1959: 19–20)

Kant thinks that the same is true of geometrical truths, e.g. that a straight line is the shortest distance between two points:

> That the straight line between two points is the shortest, is a synthetic proposition. For my concept of *straight* contains nothing of quantity,

> but only of quality. The concept of the shortest is wholly an addition, and cannot be derived, through any process of analogy, from the concept of the straight line. (1929 B16: 53)

And, as already indicated in the penultimate passage quoted, he thinks that the key notion to be appealed to in explaining how such synthetic *a priori* knowledge is possible is that of *intuition*.

An intuition for Kant is a singular representation of an object, a concept is a general representation. Concepts are the products of the understanding, to which individual representations are never given. So:

> Our nature is so constituted that our *intuition* can never be other than sensible; that is, it contains only the mode in which we are affected by objects. . . . Without sensibility no object would be given to us. (Kant 1929 A51/B75: 93)

However, Kant thinks, as well as empirical intuitions of the kind an empiricist such as Hume would recognize, we must also recognize pure intuitions which underlie our *a priori* knowledge of arithmetic and geometry. These pure intuitions constitute the *forms* supplied by the human mind, in which the *matter* of any empirical intuition must be given to us. And of these forms he says:

> there are two forms of sensible intuition, serving as principles of *a priori* knowledge, namely, space and time. (1929 A22/B63: 67)

Thus, Kant thinks, any sense experience we have of the world must conform to these forms of intuition. Any experience of outer sense (of objects other than ourselves) must conform to the form of space, and any experience at all, whether it presents itself as experience of something other than its subject or not, must conform to the form of time. Russell's famous analogy is of a man who because he wears blue spectacles sees everything blue (Russell 1946: 734). Using a similar analogy Bennett (1966: 15–16) compares the pure intuition of space, the form of outer sense, to the invariant form imposed on a piece of music by its being played on a piano. Whatever the world is like in itself, when it 'plays' on our sensibility, because of the nature of that 'instrument' the product must invariably have a spatio-temporal form. To rephrase the point in terms of Russell's analogy, we all wear spatio-temporal spectacles and so are constrained to perceive the world, however it is in itself, as spatio-temporal.

Kant finds in this doctrine an explanation of the synthetic *a priori* character of arithmetic and geometry (which he takes to be Euclidean geometry). As he explains in the *Prolegomenon*:

> But we find that all mathematical knowledge has this peculiarity, that it must first exhibit its concept *in intuition*, and do so *a priori*, in an intuition that is not empirical but pure: without this means mathematics cannot make a single step. Its judgements are therefore always intuitional. (1959: 36)

Kant illustrates the involvement of intuition in our mathematical judgements with the example of the proof that the angles of a triangle sum to two right angles. He maintains that no analysis of the concept of a triangle could ever yield this knowledge. Rather to arrive at it we must *construct* a triangle in intuition (draw one on paper or visualize one in the mind's eye) and then deduce the conclusion from the universal conditions governing the construction of triangles. In this fashion the mathematician arrives at his conclusions through a chain of inferences guided throughout by intuition (1929: 742–5).

It is the same with our arithmetical knowledge, Kant thinks. Both spatial intuition, as we saw earlier in the example of 7 + 5 = 12, in the form of intuitions of fingers or points, and temporal intuition, in the form of counting, are involved in our acquisition of arithmetical knowledge (though Kant thinks that arithmetic is particularly related to the pure intuition of time as geometry is to the pure intuition of space). Thus we see that everything we can experience must conform to the forms of time and space, and as the features of these are spelled out in arithmetic and geometry, everything we experience must conform to the rules of arithmetic and (Euclidean) geometry. A world conforming to a different geometry or a different arithmetic is not *inconceivable* – it would involve no contradiction – but it is *unimaginable*, and we can therefore know in advance that no such world could be given to us in sensibility.

Crucial to this Kantian theory, then, is the assumption that our knowledge of both arithmetic and geometry depend upon our knowledge of space and time, and that in this respect geometry and arithmetic are epistemologically on a par.

The discovery of non-Euclidean geometries and the proof of their consistency relative to Euclidean geometry thus created a serious problem for this unified Kantian theory of mathematics in the absence of consistent alternative arithmetics (see also Detlefsen 1995, to which the following is indebted).

In the work of Gauss, Lobatchevsky, Bolyai and Riemann, non-Euclidean geometry was developed by replacing the Euclidean fifth axiom, the axiom of parallels (that through any point outside a straight line there is one and only one straight line coplanar with it, which does not intersect the given straight line in either direction) with an alternative axiom. When it was established that such alternative geometries were consistent if Euclidean geometry was consistent it became tempting to conclude that we do not, after all, have *a priori* knowledge of the truth of Euclidean geometry as we have of arithmetic, but can only know it *a posteriori* on the basis of empirical intuition. Thus Gauss wrote in 1821:

> My innermost conviction is that geometry has a completely different position in our *a priori* knowledge than arithmetic . . . we must humbly admit that, although number is purely a product of our intellect, space also possesses a reality outside the mind to which we cannot ascribe its laws *a priori*. (Gauss 1863–1903: vol. 8, 200)

And later in a letter to Bolyai's father he wrote:

> It is precisely in the impossibility of deciding *a priori* between [Euclidean geometry] and [the younger Bolyai's non-Euclidean geometry] that we have the clearest proof that Kant was wrong to claim that space is only the form of our intuition. (Gauss 1863–1903: vol. 8, 220–1)

As Detlefsen explained, there thus developed a belief among nineteenth-century thinkers that arithmetic was *not* epistemologically on a par with geometry, as Kant had thought, and that a philosophy of mathematics was required which recognized this fact.

The third feature of the nineteenth-century mathematical context which was importantly present in the background of Frege's thought was the reductive programme known as 'the arithmetization of analysis', that is the programme of definition of the real numbers in terms of rational numbers (and set theory) and thus, since rational numbers can easily be defined in terms of natural numbers, in terms of natural numbers. The arithmetization of analysis was actually the culmination of a movement to introduce greater rigour in the development of mathematical analysis and particularly to define the crucial notion of a 'limit' without appeal to the paradoxical notion of infinitesimals. This movement eventually led to a focus on real numbers, which were shown to be definable (by Weierstrass, Cantor and Dedekind) purely in terms of rational numbers without any appeal to geometrical notions (which had been the earlier basis of the account of real numbers).

Dedekind stresses the need to provide a non-geometrical foundation for analysis in his 'Continuity and Irrational Numbers' (1872):

> In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such recourse to geometric intuition in a first presentation of the differential calculus, I regard as extremely useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetical and perfectly rigorous foundation for the principles of infinitesimal analysis. (1909a: 1–2)

Given his achievement it seemed to him that he had reduced the whole of analysis to a study of natural numbers. As he puts it himself in 'The Nature and Meaning of Number' (1888):

> every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers, – a declaration I have heard repeatedly from the lips of Dirichlet. (Dedekind 1909b: 135)

Given that this was so, as it seemed obvious that it was in the 1870s, the natural next question to ask was about the natural numbers themselves. And Dedekind himself made this transition, putting forward independently of Frege a version of logicism which he states thus:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of

> the notions or intuitions of space and time. That I consider it an immediate result from the laws of thought. . . . It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number domain that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our minds. (1909b: 31–2)

With this philosophical and mathematical context in mind, let us now return to the development of Frege's own thought.

From the start Frege's work had an epistemological motive and was set against a Kantian background. He begins his doctoral dissertation by insisting that geometry rests on intuition.

By contrast, he insists in his Habilitationsschrift:

> It is quite clear that there can be no intuition of so pervasive and abstract a concept as that of magnitude. There is therefore a remarkable difference between geometry and arithmetic concerning the way in which their basic laws are grounded. The elements of all geometrical constructions are intuitions, and geometry refers to intuition as the source of its axioms. Because the object of arithmetic is not intuitable, it follows that its basic laws cannot be based on intuition. (Frege 1984: 57)

The same point, that the object of arithmetic is not intuitable, is made in section 105 of the *Foundations*:

> In arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it. (Frege 1968)

Frege attempted to establish a philosophy of mathematics which respected what he took to be the fundamental difference, demonstrated by the existence of non-Euclidean geometries, between geometry and arithmetic: that arithmetic, in contrast to geometry, is not merely applicable to everything that is intuitable, but to everything that is numerable, that is, to everything that is *conceivable*.

This generality of arithmetic is something he stresses in section 14 of *Foundations* where his recognition of the consistency of non-Euclidean geometries is also indicated:

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially

> intuitable, whether natural or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and trees turn into men, whence the drowning haul themselves up by their own topknots – all these remain, so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions is not useless, by any means, but it is to leave the ground of intuition entirely behind. If we do make use of intuition even here, as an aid, it is still the same old intuition of Euclidean space, the only space of which we have any picture. Only then the intuition is not taken at its face value, but as symbolic of something else; for example, we call straight or plane what we actually intuit as curved. For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradiction when we proceed to our deductions, despite the conflict between our assumptions and our intuitions. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic and consequently are synthetic. Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying one of them, and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number then, be connected very intimately with the laws of thought? (Frege 1968: 20–1)

To explain the universal validity of arithmetic was the fundamental motive behind Frege's philosophy. To do so he believed he needed to establish the independence of arithmetic from intuition, and hence geometry, and to do that, in turn, he needed to establish that at no point did intuition need to be appealed to in arithmetical proof (see also Demopoulos 1994, to which the following is greatly indebted). It was for this purpose, he explains in *Foundations*, that he invented his conceptual notation (*Begriffsschrift*), to give 'gapless' mathematical proofs:

> In proofs as we know them, progress is by jumps, which is why the variety of types of inference in mathematics appears to be so excessively rich . . . the correctness of such a transition is immediately self-

> evident to us . . . whereupon, since it does not obviously conform to any of the recognized types of logical inference, we are prepared to accept its self-evidence forthwith as intuitive, and the conclusion itself as a synthetic truth – and this even when obviously it holds good of much more than merely what can be intuited.
>
> On these lines what is synthetic and based on intuition cannot be sharply separated from what is analytic . . .
>
> To minimize these drawbacks, I invented my conceptual notation. It is designed to produce expressions which are shorter and easier to take in, . . . so that no step is permitted which does not conform to the rules which are laid down once and for all. It is impossible, therefore, for any premiss to creep into a proof without being noticed. In this way I have, without borrowing any axiom from intuition, given a proof of a proposition which might at first sight be taken for synthetic which I shall here formulate as follows:
>
>> If the relation of every member of a series to its successor is one- or many-one, and if m and y follows in that series after x, then either y comes in that series before m, or it coincides with m, or it follows after m. (1968: 102–3)

This proposition is number 133 in *Conceptual Notation*, to the proof of which Frege devotes the whole of its part III. In his introductory comments in this part of *Conceptual Notation* he once again stresses the way in which intuition can be shown by the use of his conceptual notation to be inessential to arithmetical proof:

> Throughout the present example [the proof of 133] we see how pure thought, irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgements that at first sight appear to be possible only on the basis of some intuition. (1972: 167)

Alerted by this, the significance of Frege's apparently casual reference to 'something intuitive' in the Preface to *Conceptual Notation*, in which he explains how the idea of his conceptual notation arose, will not be missed:

> we divide all truths which require a proof into two kinds: the proof of the first kind can proceed purely logically, while that of the second kind must be supported by empirical facts . . . not the psychological mode of origin, but the most perfect method of proof underlies the classification.
>
> Now while considering the question to which of these two kinds do judgements of arithmetic belong, I had first to test how far one

> would get in arithmetic by means of logical deductions alone, supported only by the laws of thought, which transcend all particulars. The procedure in this effort was this: I sought first to reduce the concept of ordering-in-a-sequence to the notion of *logical* ordering, in order to advance from here to the concept of number. So that something intuitive could not squeeze in unnoticed here, it was important to keep the chain of reasoning free of gaps. As I endeavoured to fulfil this requirement most rigorously, I found an obstacle in the inadequacy of the language; despite all the unwieldiness of the expressions, the more complex the relations became, the less precision – which my purpose required – could be obtained. From this deficiency arose the idea of the 'conceptual notation' presented here. Thus, its chief purpose should be to test in the most reliable manner the validity of a chain of reasoning and expose each presupposition which tends to creep in unnoticed, so that its source can be investigated. (1972: 103–4)

So far I have been stressing the anti-Kantian motivation of Frege's work: his desire to demonstrate that arithmetic, in contrast to Euclidean geometry, is applicable to everything conceivable and does not require any ground in intuition. The link between Frege's thought and the arithmetization of analysis is related and can be better understood with this fundamental motivation in mind. As we saw in the passages quoted from Dedekind, the separation of analysis from geometry and the definition of the notion of a real number without appeal to geometrical concepts was one of the chief aims of this programme. And in extending this programme to the natural numbers, and thus putting forward his own version of logicism, Dedekind again stresses that he considers 'the number concept entirely independent of the notion or intuitions of space and time'. In fact, this is part of what 'logicism' means for Dedekind: 'In speaking of arithmetic (algebra, analysis) as part of logic I *mean to imply* that I consider the number-concept entirely independent of the notions or intuition of space and time' (1909b: 31, my italics). Logicism is thus not defined positively, in terms of what logic includes, but rather negatively, in terms of what it uncontroversially excludes – Kantian intuition.

Thus Frege's logicism was not a novel addition to nineteenth-century mathematics or philosophy, but rather belonged to an accepted tradition and had a comprehensible motivation in the context of its time. Later, particularly in the writings of the logical empiricists of the Vienna Circle, logicism came to be thought of as a way of demonstrating that the Kantian synthetic *a priori* need not be acknowledged at all, even in the domain of mathematics, in which it seemed most plausible to apply it. But such a *global* anti-Kantianism was never Frege's position: as we have seen, he remained wedded throughout his life to the synthetic *a priori* character of Euclidean geometry.

So far we have been considering the original motive of Frege's work. This remained constant until the time, after he became convinced that Russell's Paradox presented an insuperable roadblock (probably in 1906), that he abandoned his fundamental interests (only resuming them in the last years of his life, when he finally came to accept the Kantian view he had for so long opposed). However, Frege's thought nonetheless underwent important changes and developments during this time.

Initially, in *Conceptual Notation* Frege's aim appears to have been the relatively modest one of showing that *some* mathematical propositions, such as his number 133, which had previously been thought to be establishable only by an appeal to Kantian intuition, could be given gap-free proofs in his logical system and thus demonstrated without appeal to intuition. In this way the aim of *Conceptual Notation* was to establish that Kant's view of mathematics was mistaken, since that view implied that intuition was indispensable in (at least some) cases where it could be proved that this was not so. But either during the writing of *Conceptual Notation*, or soon after, Frege became convinced that intuition could be seen to be unnecessary *throughout* arithmetic and that the latter could be given a wholly logical foundation. Thus he announces at the end of its Preface:

> arithmetic was the point of departure for the train of thought that led me to my ideography. And that is why I intend to apply it first of all to that science, attempting to provide a more detailed analysis of the concepts of arithmetic and a deeper foundation for its theorems. For the present I have reported in the third chapter some of the developments in this direction. To proceed further along the path indicated, to elucidate the concepts of number, magnitude and so forth – all this will be the object of further investigations, which I shall publish immediately after this booklet. (1972: 107)

The 'further investigations' here referred to, of course, appeared, after a lengthy delay, in *Foundations*. The logic, philosophical logic and theory of meaning of the *Foundations* are those of *Conceptual Notation* applied to the development of Frege's logicist thesis.

After its publication, however, important developments took place in Frege's system. *Conceptual Notation* and *Foundations* operated with a notion of the conceptual content of a sentence or subsentential expression. In his article 'On Sense and Reference' Frege distinguished within this notion the two notions of *Sinn* (sense) and *Bedeutung* (reference) (in Frege 1969). This distinction became central to Frege's thought thereafter and is perhaps his most important contribution to philosophy.

Secondly, in his article 'Function and Concept' (also in Frege 1969) Frege develops and makes explicit the identification of concepts with functions mapping arguments on to truth-values which then becomes incorporated into his formal treatment of arithmetic in *Basic Laws*. In *Conceptual Notation* Frege operates with the notion of a function, but his explicit definition of what a function is restricts it to linguistic expressions, and in *Foundations*, though the notion of a concept is central to Frege's argument – one of his fundamental contentions being that a statement of number is an assertion *about* a concept – he does not give the notion any analysis or identify concepts with functions.

Associated with Frege's distinction between sense and reference and his identification of concepts with functions is another doctrine Frege introduced after *Foundations* – the doctrine that sentences are proper names of truth-values. Since Frege had now distinguished the earlier notion of conceptual content into the two components of sense and reference, and since he applied this distinction across the board, he was led to distinguish between the sense of a complete sentence (which he called a 'thought') and its reference. Regarding concepts (the reference of predicates) as functions he was then led to the conclusion that sentences, as a particular kind of completed complex functional expression, must have as their references truth-values.

Also associated with the other doctrines introduced by Frege after *Foundations* was his definite adherence to the identification of numbers with the extensions of concepts (classes). In *Foundations* Frege's concern had been to explain how numbers, as objects, could be given to us otherwise than in intuition. He was led to identify them with extensions of concepts, taken to be logical objects already understood, by a line of thought we will examine later. But he is quite explicit that this identification is tentative and not central. After *Foundations*, given the explicit identification of concepts with functions, he is able to explain extensions of concepts as a special case of value ranges of functions – objects defined by his Basic Law (V) – and commits himself to the identification.

Apart from these changes Frege's system remains the same after *Foundations* and in the prolegomenon to *Basic Laws* he gives a precise statement of the philosophical logic developed after *Foundations* in terms of which he then proceeds to the development of the gapless system of proofs required to demonstrate his logicism.

After the discovery of the vulnerability of his system to Russell's Paradox the one departure he makes is the rejection of value ranges as fictions of our language: the distinction between sense and reference, the characterization of concepts as functions and all the associated elements of the system are retained.

Frege's Contributions to Philosophy

Frege's contributions to philosophy fall into four areas – logic, philosophy of mathematics, philosophical logic and the theory of meaning – and it will be convenient now to summarize them.

Frege's definitive contributions in logic, as we have noted, were already present in *Conceptual Notation*. Logic before Frege had been dominated by the Aristotelian theory of the syllogism, which was concerned with the validity of inferences involving general sentences, that is those containing such expressions as 'every', 'some', 'no' and their synonyms. An example of a syllogism is:

Every swan is an animal
Every animal is a living thing
Ergo: Every swan is a living thing

In each of these sentences there is a subject term (in the first sentence this is 'swan') and a predicate term ('animal' in the first sentence). The term which occurs in both premisses ('animal') is called the 'middle term'.

Thus Aristotelian logic is a logic of *terms*. It is concerned with the various patterns of argument which can be exhibited by combining the expressions 'every', 'some', 'no' and their synonyms with terms. For example, the pattern exhibited by the above argument is 'A belongs to all B and B belongs to all C therefore A belongs to all C' in which 'A', 'B' and 'C' stand in for any terms whatever.

A competing logical theory, developed by the Stoics, was a logic of sentences, or propositions. The Stoics were interested in the validity of arguments like the following:

If it is day then it is light
It is day
Ergo: It is light.

The pattern this exhibits, in virtue of which the argument is valid, is expressed as follows by the Stoics: 'If the first then the second, the first, therefore, the second.' Thus Stoic logic was concerned with the validity of those patterns of argument which can be exhibited by replacing sentences occurring in more complex sentences by symbols ('the first', 'the second') which stand in for any sentence whatever. The patterns the Stoics studied were those whose validity depended on the occurrence in them of the sentential operators: negation ('It is not the case that'), conjunction ('Both . . . and . . . '), disjunction ('Either . . . or . . . ') and the conditional ('If . . . then . . . '), which are used to construct more complex sentences from simpler ones.

These two theories of logic were merged in the work of George Boole (1815–1864), who developed an algebraic system whose formulae, differently interpreted, could be taken either as expressing the general propositions which were the subject matter of the Aristotelian logic of terms or the complex propositions which were the subject matter of Stoic logic. Boole called propositions of the former type 'primary propositions' and propositions of the latter type 'secondary propositions'. This indicated his view of their relative priority. He thought that secondary propositions could be understood as generalizations about classes of occasions or times and so could be reduced to primary propositions.

The greatest weakness of logic as developed before Frege was its inability to deal with sentences containing expressions for *multiple generality*, sentences like 'Every boy loves some girl', or 'some girl is admired by every boy', in which two or more general expressions ('every boy', 'some girl') are joined by an expression for a relation. Such sentences, which are, of course, commonplace in mathematics (e.g. 'Every even number is the sum of two primes'), are frequently ambiguous. Medieval logicians elaborated theories of 'supposition', in part to deal with multiple generality and explain the ambiguities, but could not provide a convincing account.

In addition, even in Boole's system, in which both the traditional Aristotelian logic of terms and the Stoic logic of propositions can be represented, the validity of inferences from the one type of proposition to the other cannot be represented – precisely because for

Boole term logic and propositional logic are *two* interpretations of the *same* algebraic system.

The logical system Frege introduced in *Conceptual Notation* resolved these difficulties once and for all. Its key was the replacement of the *grammatical* notions of subject and predicate, central to Aristotelian syllogistic, by the *mathematical* notions of argument and function.

Thus Frege viewed the sentence:

> Carbon dioxide is heavier than hydrogen

not as asserting of the subject 'carbon dioxide' the predicate 'is heavier than hydrogen' but as the value of the function 'is heavier than hydrogen' for the argument 'carbon dioxide', or again as the value of the function 'carbon dioxide is heavier than' for the argument 'hydrogen', or again as the value of the third function 'is heavier than' for the pair of arguments 'carbon dioxide' and 'hydrogen'. Analogously, 2 + 3 (that is, the number 5) can be regarded as the value of the function designated by '( ) + 3' for the argument 2, or as the value of the different function designated by '2 + ( )' for the argument 3, or as the value of the third function designated by '( ) + ( )' for the pair of arguments 2 and 3.

On the basis of this innovation Frege was then able to introduce his most important logical discovery – the quantifier. We can express the fact that carbon dioxide is heavier than hydrogen, using Frege's function/argument language, in some such way as follows:

> The function 'is heavier than hydrogen' is a fact for the argument 'carbon dioxide'

And, since hydrogen is the lightest gas (and we are only considering gases):

> The function 'is heavier than hydrogen' is a fact whatever we take as its argument

This, of course, states the general claim that *everything* is heavier than hydrogen. Using now familiar logical symbolism, which is not different in essentials from Frege's, we can express this as follows: (∀x)(x is heavier than hydrogen). Here the *universal quantifier* '(∀x)', captures the generality of Frege's phrase 'whatever we take as its argument'.

As we shall see in more detail later, the introduction of the quantifier (and its associated variable) enables Frege to express unambiguously, in a perspicuous notation, not only singly general propositions like the one in the example, but also multiply general propositions like 'Every gas is heavier than some gas'. When combined with Frege's innovations in propositional logic (the branch of logic studied by the Stoics), it also enables him to express unambiguously in his symbolism *every* proposition expressible in Aristotelian syllogistic and *every* multiply general proposition no matter how complex. The essential insight which made this possible for Frege was his reversal in the order of explanatory priority of Boole's 'primary' and 'secondary' propositions. (It is this reversal Frege expresses by his repeated statement that, unlike other logicians, he proceeds from judgements rather than concepts.)

Frege's other great contribution to logic in *Conceptual Notation* was his construction of the first formal system. A formal system has three parts: a precisely specified language in which the propositions of the system can be expressed, a set of axioms, and a specification of rules by which theorems can be deduced from the axioms in accordance with purely formal or syntactic constraints (i.e. on the basis of their shapes). The formulae of the formal system will express particular propositions under the intended interpretation, but it will be unnecessary to know what propositions are expressed by the formulae in a derivation in order to check whether that derivation is in accordance with the rules. That can be done mechanically simply by attending to the shapes of the formulae. Frege's system as presented in *Conceptual Notation* is not quite flawless, since it employs a rule of inference (a rule of substitution) which is not explicitly stated, but this defect is remedied in *Basic Laws*. Given a formal system it makes sense to ask whether it is complete, whether all the logical truths it is intended to capture can indeed be derived from its axioms via its rules. Frege's formulation in *Conceptual Notation* (with the implicit rule of substitution made explicit) is a complete axiomatization of first-order logic and contains a complete axiomatization of propositional logic as a subsystem.

In the philosophy of mathematics, as we have already seen, Frege's primary aim was to establish the independence of arithmetic from intuition and thus to refute Kant's contention that it is synthetic *a priori*. In *Foundations* he consequently expresses the thesis he wishes to defend in the Kantian terminology as the thesis that arithmetic is analytic. But he is equally concerned to refute the empiricism of John Stuart Mill, whose position, in Kantian terms, is that arithmetic is synthetic *a posteriori*. Thus he devotes the first part of *Foundations* to a critique of these two philosophies and elaborations of them. His criticisms, which we shall examine later, are devastating, and enable him to proceed to his positive contentions from the firmly established basis of several conclusions about what numbers are *not*: they are not subjective ideas, nor physical objects, nor collections or properties of such. Frege's first and most famous positive contention he expresses in his claim: 'The content of a statement of number is an assertion about a concept.'

If I say, for example, 'Venus has 0 moons', I am not making a statement about the moons of Venus (which do not exist if I am right). Rather I am assigning a property to the concept *moon of Venus*. I am saying how many things there are falling under that concept. Another way of putting this fundamental point is to say that a statement of number is a statement about a *kind of thing* – a statement about how many things there are of that kind.

It is an important implication of this point that the fundamental method of referring to numbers is via descriptions of the form 'the number belonging to the concept F' or, more briefly, 'the number of F's': for example, 'the number of moons of Venus'. This apparently trivial linguistic claim is crucial for Frege, because it enables him to provide an answer other than the Kantian one to Kant's question: 'How are numbers given to us?'

For Kant the only possible answer to this question is: they are given to us in intuition. For numbers, he thinks, are objects and objects can only be given in intuition. Frege accepts the Kantian distinction between objects and concepts, and indeed states as a fundamental principle in *Foundations* 'never to lose sight of the distinction between concept and object', but he also insists that numbers are objects. Thus he is faced with the need to explain how numbers are given to us 'if we cannot have any ideas or intuitions of them' (1968: 73). His answer appeals to the second fundamental principle he states at the outset of the *Foundations*, his famous 'context principle': 'never to ask for the meaning of a word in isolation but only in the context of a sentence.' The precise meaning and status of this principle in Frege's philosophy has become the most controversial question in Fregean scholarship. But its significance for Frege at the point in his discussion we are concerned with is that it enables him to make what Michael Dummett has called 'the linguistic turn' (Dummett 1991a: 111–12): to convert what is plainly for Kant an epistemological question into a question about language. 'Since', Frege writes, 'it is only in the context of a sentence that words have any meaning, our problem becomes this: to define the sense of a sentence in which a number word occurs.'

Since numbers are objects, Frege now goes on to say, the most important form of sentence whose sense must be explained is one asserting identity: 'if we are to use the symbol *a* to signify an object, we must have a criterion for deciding in all cases whether *b* is the same as *a*' (1968: 73). But given that the fundamental way to refer to a number is as the number associated with a certain concept, it follows that what needs explaining is the sense of sentences of the form:

> the number which belongs to the concept F is the same as that which belongs to the concept G

To explain this is to give the *criterion of identity* for numbers.

The criterion Frege proposes is *one–one correlation*. A one–one correlation holds between the F's and the G's if and only if every F can be paired with exactly one G and every G with exactly one F so that no F or G is left over. Frege is able, appealing to the treatment of relations in *Conceptual Notation*, to give a purely logical definition of one–one correlation, and so is able to put forward what has now come to be called Hume's Principle (because Frege introduces it with a quotation from Hume):

> The number of F's is identical with the number of G's if and only if there is a one–one correlation between the F's and G's

as a definition of numerical identity in purely logical terms.

It is at this point that things start to get murky. For Frege now rejects Hume's Principle as an adequate account of numerical identity on the ground that it does not allow us to determine whether Julius Caesar is a number! This objection has come to be known as 'the Julius Caesar objection'. Nevertheless Frege does not dismiss Hume's Principle completely but treats its derivability as a criterion of adequacy of the explicit definition of number he goes on to give in terms of extensions of concepts (classes). However, recent work on Frege's philosophy of arithmetic has made it clear that the role of Hume's Principle is arguably more significant than this suggests. For first, it can be proved that if Hume's Principle is added as the sole additional axiom to second-order logic, that is, the logic of *Conceptual Notation*, then axioms sufficient for arithmetic (Peano's axioms or Frege's own set, which are equivalent) can be derived (see Parsons 1965; Wright 1983; Boolos 1998). This thesis has come to be known, following a proposal made by George Boolos, as 'Frege's Theorem'. Secondly, it can be seen by attention to the actual structure of Frege's informal proof sketches in *Foundations* and his formal proofs in *Basic Laws* that Frege only ever makes essential appeal to his explicit definition of number in terms of extensions of concepts to derive Hume's Principle. Everything else is derived from Hume's Principle. Thus Frege himself proved Frege's Theorem (Boolos 1998; Heck 1993).

The importance of these mathematical and historical facts is disputed. Some modern commentators (notably Crispin Wright (1983 and 1998) and Bob Hale (1994)) take them as implying that, despite Russell's Paradox, Frege's essential insights can be defended: Hume's Principle can be taken as explanatory of our concept of number and since its addition to second-order logic is all that is required for arithmetic, arithmetic can be seen, not indeed merely as logic, but at least as a body of analytic truths. Other commentators, notably Michael Dummett (1991a) and George Boolos, deny this, and take it that Frege's logical project must be seen as a, doubtless magnificent, failure. One major issue in this debate is whether the Julius Caesar problem, which caused Frege to move to his explicit identification of numbers with the extensions of concepts, can be solved without taking this step, or legitimately ignored.

Two more of Frege's achievements in the philosophy of mathematics may be mentioned at this point: his definition (introduced in *Conceptual Notation*) of the notion of the ancestral of a relation (that relation which stands to the given relation as *being an ancestor of* stands to *being a parent of*) and his definition of the natural, or finite, numbers in *Foundations* as those which stand to zero in the ancestral of the relation of *immediate successor of*.

The general definition of the ancestral of a relation was very important to Frege's anti-Kantian project because it enabled him to explain the general notion of *following in a series* in purely logical terms, without any appeal to spatial or temporal intuition, and his definition of the *natural numbers* as the objects which stand to zero in the ancestral of the relation *immediate successor of*, given his definition of the ancestral in general (in terms, as we shall see, of the notion of a 'hereditary' property), enables him to establish immediately the *logical* validity of proof by induction in mathematics (since, in effect, his definition of the natural numbers is: the objects for which mathematical induction works).

Turning now to Frege's contribution to philosophical logic, first and foremost must be mentioned his functional theory of predication. We have already looked briefly at this in connection with Frege's logical contribution in *Conceptual Notation*. However, after *Conceptual Notation* Frege extends the theory and makes it much more precise. In *Conceptual Notation* the proposal is that a sentence like 'Hydrogen is lighter than carbon dioxide' may *itself* be seen as the value of a function for an argument, in fact, as we have seen, as the value of the different functions for different arguments, just as the number 2 + 3 (i.e. 5) may be seen as the value of different functions for different arguments. Thus Frege's focus at this stage is on *linguistic* items: sentences, the names they contain and the functions which we can regard as mapping names as arguments on to sentences as values. There is no explicit development of the thought that not only linguistic items but also the extra-linguistic items corresponding to them should be regarded similarly as functionally related. After *Foundations* this development takes place. Frege now suggests that in a sentence such as 'Socrates is wise' (to keep to a simple example) we can discern a proper name 'Socrates' and a 'concept word' or predicate ('is wise'). Corresponding to the name is the object for which it stands and corresponding to the 'concept word' is the concept for which it stands, and this concept is *itself* a function mapping objects onto values. In particular, just as the function designated by '( ) + 3' maps the number 2 onto the number 5, that is, the number which is designated by '2 + 3', so the concept for which '( ) is wise' stands maps the man Socrates onto that which is designated by the sentence 'Socrates is wise'. But what is designated by this sentence? Frege's answer is: the truth-value True (since the sentence *is* true). Thus he arrives at the conclusion that concepts are functions mapping arguments onto truth-values. Similarly, relations, which are analogous to the addition function, map *pairs* of arguments onto truth-values.

This functional theory of predication is fundamental to Frege's philosophy of logic; whether it can be sustained, or whether in the end it can only be seen as an illuminating analogy depends on whether Frege's view of truth-values as objects can be defended. But whichever is the case, it enables Frege to achieve a second fundamental insight which allows him to sweep aside as irrelevant thousands of years of debate about the distinction between universals and particulars. According to traditional doctrine the sentence 'Socrates is wise' introduces, via the predicate 'is wise', the universal *wisdom* into our discourse, as something said of Socrates. And *the very same item* may be introduced into our discourse, via the name 'wisdom', as something about which something else is said, for example, in the sentence 'Wisdom is a characteristic of the old'. Particulars, like Socrates, are then those items about which other things may be said but may not *themselves* be said of other things – they are the ultimate subjects of predication. Frege rejected this traditional view entirely. For he maintained that concepts are essentially predicative. What the predicate (or as he prefers to say, concept-word) '( ) is wise' stands for is a concept, and as such cannot be the reference of a proper name like 'Wisdom'. 'Wisdom' is a perfectly good proper name, and there is no need to deny it a reference, but its reference cannot be the same as that of any predicate, even the predicate '( ) is wise'.

The reason for this is that predicates, unlike proper names, are essentially incomplete. A predicate must be written with an indication of the gap(s) to be filled, not as a matter of convention, but because a predicate is not itself a quotable bit of a sentence, as a proper name is, but a pattern exhibited by the sentences which contain it. Thus proper names and predicates are of a fundamentally different character and simply could not play the same linguistic role. And correspondingly, Frege maintains, concepts and objects are ontologically of a fundamentally different character. No concept could be an object, because no concept could be the reference of a proper name.

Given Frege's functional theory of predication, the incompleteness of concepts emerges straightforwardly as a special case of the, perhaps at first sight more intuitively appealing, incompleteness of functions. That a function, as a mapping from objects or functions to objects, is itself awaiting completion, or saturation, by arguments has an initial appeal that the idea of the essential incompleteness of concepts (thought of as the references of concept-words or predicates) does not possess. Thus Frege's espousal of the functional theory of predication made it easy for him to accept the incompleteness, and therefore essentially predicative nature, of concepts. However, Frege's arguments for the incompleteness of concepts do not depend upon the functional theory of predication. What is essential is rather Frege's insight, expressed in his rejection of the traditional doctrine of subject and predicate, that the very same sentence can be analysed equally legitimately as saying different things of (the same or different) things, and that certain sentences, on certain analyses, need not contain any quotable part corresponding to what is said. Thus, to use Frege's own example, what 'Cato killed

Cato' says of Cato when broken up into 'Cato' and '( ) killed ( )' does not correspond to any quotable part of the sentence, but rather to its exhibiting the pattern of containing the verb 'killed' preceded and followed by two occurrences of the *same proper name*: thus we say the very same thing of Socrates with the sentence 'Socrates killed Socrates'.

Frege's insistence on the essentially predicative nature of a concept leads him into paradox, however. For he is led to deny that the concept *horse* is a concept – since 'the concept *horse*' is not an incomplete expression. This 'awkwardness of language', as Frege calls it, has been much debated. But it seems clear that Frege's conclusion (at least in 'Concept and Object') is that what we must recognize here is that there are fundamental ontological divisions in the world corresponding to fundamental linguistic divisions, and that these divisions cannot be put into words – cannot be stated, but are recognized implicitly by any competent language-user.

Finally, in this brief look at Frege's fundamental contributions, we must now turn to his theory of meaning; and first of all, of course, to his argument for his distinction between sense and reference.

In *Conceptual Notation* and *Foundations* Frege operated with an undifferentiated notion of the 'content' of a linguistic expression. But in 'On Sense and Reference' he distinguishes within this notion the two notions of sense and reference. His argument for this distinction starts from the puzzle of identity: the fact that identity statements can be both true and informative. Thus 'The Morning Star is the Evening Star' can convey new information to someone who has not heard it before. 'The Morning Star is the Morning Star' cannot.

Yet both 'the Morning Star' and 'the Evening Star' stand for the same thing, the planet Venus. If we consider only what the expressions in the two identity statements stand for, therefore, we cannot account for the difference in potential informativeness – or 'cognitive value' – of the two. Frege's solution is to say that associated with each of the two names 'the Morning Star' and 'the Evening Star' there is, as well as a reference, a sense – which is what someone understands when they grasp the expression – and the difference in cognitive value is accounted for by the different senses of the two names.

Once Frege has introduced the distinction between sense and reference in this way he applies it generally, not only to singular terms, but also to predicates and to sentences. In the case of predicates he insists on distinguishing the concept which is the reference of the predicate both from its extension – the class of objects it applies to – and from its sense. The former distinction is made on the grounds that concepts are incomplete whereas extensions are complete, and the latter on the grounds that concepts, unlike senses, have extensional identity conditions – two predicates, for example '( ) is a creature with a heart' and '( ) is a creature with kidneys', which differ in sense, because what it is to grasp one differs from what it is to grasp the other, might apply (as in this case) to exactly the same items and thus will stand for the same concept. In the case of sentences, Frege again makes the distinction between sense and reference – introducing the name 'thought' for the sense of a sentence and arguing that the reference of a sentence must be regarded as its truth-value. A key premiss in his argument for this conclusion is his compositionality principle for reference: that the reference of a complex expression is determined by the references of its components. Frege is able to argue powerfully that nothing else can be the reference of a sentence since nothing else remains unchanged when component expressions are replaced by others with the same reference.

The distinction between sense and reference is initially appealing but it has, in fact, become a focus of intense debate and its tenability and precise nature are a matter of considerable controversy. We shall be exploring the reasons for this in the last chapter of the book. But now we must return to look with more care at Frege's logic, which is the foundation on which all of his work rests.